# Mathematical and numerical analysis of a fully coupled Stokes–Biot–transport model

SERGIO CAUCAO* XING WANG† IVAN YOTOV‡

June 18, 2026

**Abstract**

We develop a miscible displacement model for the coupling of fluid–poroelastic structure interaction (FPSI) with transport. The FPSI model couples the Stokes equations for the free fluid with the Biot system of poroelasticity via dynamic and kinematic interface conditions. The FPSI system is coupled with an advection–diffusion equation for the transport of a substance within the fluid. The model is two-way coupled and nonlinear, with concentration-dependent fluid viscosity and velocity-dependent advection and diffusion–dispersion. We establish existence and uniqueness of a weak solution via analysis of the decoupled subproblems and a fixed-point iteration argument. We then consider the finite element approximation of the model and obtain stability and error estimates for the semi-discrete continuous-in-time formulation. Computational experiments are presented to verify the theoretical convergence rates and to illustrate the feasibility of the method to simulate physically realistic transport in channelized poroelastic media.

## 1 Introduction

The interaction of a free fluid with an adjacent poroelastic medium, referred to as fluid–poroelastic structure interaction (FPSI), has a wide range of applications, including coupled surface–subsurface hydrological systems, hydrocarbon production in fractured reservoirs, design of industrial filters, arterial flows, and modeling of organs, such as the lung and the brain. The model couples the Stokes or the Navier–Stokes equations for the free fluid flow with the Biot system of poroelasticity via dynamic and kinematic interface conditions. The system exhibits features of both coupled free fluid–porous media flows [4, 31, 33, 39, 42, 46, 53] and fluid–structure interaction [6, 17, 45]. There is a growing literature on the mathematical and numerical modeling of the FPSI problem in recent years. For mathematical analysis we refer the reader to [1, 7, 20, 21, 43, 44, 49, 52, 55]. Analysis of various discretization methods, splitting methods, and preconditioning techniques can be found in [1, 3, 5, 9, 14–16, 19, 20, 22–24, 30, 43, 44, 49, 51, 55].

On the the other hand, many applications involve the coupling of the FPSI system with the transport of a substance within the fluid, such as contaminant tracking and cleanup in hydrological systems, drug delivery and low-density lipoproteins (LDL) transport in blood flows, and proppant

*Departamento de Matemática y Física Aplicadas and Grupo de Investigación en Análisis Numérico y Cálculo Científico (GIANuC$^2$), Universidad Católica de la Santísima Concepción, Casilla 297, Concepción, Chile, email: scaucao@ucsc.cl. Supported in part by ANID-Chile through the projects CENTRO DE MODELAMIENTO MATEMÁTICO (FB210005) and Fondecyt 1250937.

†Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA, email: xiw117@pitt.edu.

‡Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA, email: yotov@math.pitt.edu.
Partially supported by the US National Science Foundation grant DMS-2410686 and the Alexander von Humboldt Foundation via the Humboldt Research Award.

injection in hydraulic fracturing. The FPSI–transport problem has been significantly less studied. A one-way coupled Stokes–Biot–transport model was first developed and analyzed in [2]. One-way coupled FPSI–transport models are employed in [54] for simulations of drug-eluting stents and in [13] for modeling bioartificial organ scaffolds. To the best of our knowledge, two-way coupled FPSI–transport models have not been studied in the literature.

In this paper, we develop a nonlinear two-way coupled Stokes–Biot–transport model based on the classical miscible displacement model [32,35]. We consider the quasistatic Stokes–Biot problem and employ a variational formulation using velocity–pressure for Stokes flow, displacement for elasticity, and velocity–pressure for Darcy flow as the main variables. Similar to [3], a pressure/stress Lagrange multiplier is utilized to impose weakly the continuity of normal velocity, which is essential in this formulation. The Stokes–Biot system is coupled with an advection-diffusion transport equation for the concentration of a substance within the fluid. The system is fully coupled, with the fluid viscosity depending on the concentration and the convective and diffusion–dispersion terms in the transport equation depending on the fluid velocity. For the analysis of related two-way coupled Stokes–Darcy–transport or Navier–Stokes–Darcy–transport models, we refer the reader to [25,26,48,56]. Two-way coupled non-isothermal Biot–transport models have been studied in [11, 12, 18].

The main contributions of this paper are as follows. In the first part of the paper we establish existence, uniqueness, and stability of the weak solution of the coupled system, which is done in several steps. First, we consider the decoupled and linearized subproblems and employ a Galerkin approach based on space discretization to establish their well posedness. The theory of differential algebraic equations (DAEs) is utilized for the existence of the semidiscrete Stokes–Biot solution. A key ingredient is constructing compatible initial data for all variables. The theory of ordinary differential equations (ODEs) is used to obtain existence and uniqueness of a solution of the semidiscrete transport equation. Stability bounds are then established, followed by a compactness argument and passing to the limit to obtain well posedness of the continuous problems. This step requires a bound on the divergence of the Darcy velocity, which in turn is based on bounding the time derivatives of the displacement and Darcy pressure. The next step is establishing well posedness of the fully coupled nonlinear problem. For this purpose, motivated by the approach in [12], we consider an iterative algorithm, where the two subproblems are solved in an alternating fashion. We establish a contractive property of the iteration, which allows us to conclude that it converges to a solution of the coupled problem. We further establish uniqueness of the solution. In the second part of the paper we consider a semidiscrete finite element approximation to the fully coupled problem, based on stable Stokes and Darcy velocity–pressure pairs, $H^1$-conforming finite elements for the displacement and concentration, and a non-conforming finite element space for the Lagrange multiplier. We establish well posedness using the framework developed for the weak formulation and derive error estimates with convergence rates for the numerical solution. Numerical experiments using a fully discrete method with backward Euler time discretization are presented to verify the convergence rates and illustrate the performance of the method for the simulation of physically realistic transport in channelized poroelastic media.

The outline of the paper is as follows. In Section 2, we present the mathematical model and its weak formulation. Section 3 is devoted to the well posedness of the decoupled and linearized subproblems. The analysis of the fully coupled nonlinear problem based on the convergence of the iterative algorithm is presented in Section 4. The semidiscrete numerical method is introduced in Section 5, where well posedness of the method is established. The error analysis of the method is developed in Section 6. Section 7 is devoted to the numerical experiments.

## 2 The coupled model and its weak formulation

### 2.1 Stokes–Biot–transport model

Let $\Omega \subset \mathrm{R}^d$, $d \in \{2,3\}$, be a union of non-overlapping polygonal regions $\Omega_f$ and $\Omega_p$. Here $\Omega_f$ is a free fluid region with flow governed by the Stokes equations and $\Omega_p$ is a poroelastic region governed by the Biot system. Let $\Gamma_{fp} = \partial\Omega_f \cap \partial\Omega_p$ denote the (non-empty) interface between these regions and let $\Gamma_f = \partial\Omega_f \setminus \Gamma_{fp}$ and $\Gamma_p = \partial\Omega_p \setminus \Gamma_{fp}$ denote the external parts on the boundary $\partial\Omega$. We denote by $\mathbf{n}_f$ and $\mathbf{n}_p$ the unit normal vectors that point outward from $\partial\Omega_f$ and $\partial\Omega_p$, respectively, noting that $\mathbf{n}_f = -\mathbf{n}_p$ on $\Gamma_{fp}$. Let $(\mathbf{u}_\star, p_\star)$ be the velocity-pressure pair in $\Omega_\star, \star \in \{f,p\}$. Let $\boldsymbol{\eta}_p$ be the displacement in $\Omega_p$, $\mu(c)$ be the fluid viscosity that depends on the concentration $c$ of a substance of a solute transported by the fluid, $\mathbf{f}_\star$ be the body force, and $q_\star$ be external source or sink terms. In the free fluid region $\Omega_f$, the velocity $\mathbf{u}_f$ and pressure $p_f$ satisfy the Stokes equations

$$-\nabla\cdot\boldsymbol{\sigma}_f = \mathbf{f}_f\,, \quad \nabla\cdot\mathbf{u}_f = q_f \quad \text{in} \quad \Omega_f\times(0,T]\,, \tag{2.1a}$$

$$\mathbf{u}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^D\times(0,T]\,, \quad \boldsymbol{\sigma}_f\mathbf{n}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^N\times(0,T]\,, \tag{2.1b}$$

where $\boldsymbol{\sigma}_f = -p_f\mathbf{I} + 2\mu(c)\boldsymbol{\epsilon}(\mathbf{u}_f)$ is the Cauchy stress tensor, $\boldsymbol{\epsilon}(\mathbf{u}_f) = \dfrac{1}{2}\left(\nabla\mathbf{u}_f + (\nabla\mathbf{u}_f)^{\mathrm{t}}\right)$ stands for the deformation rate tensor, $T>0$ is the final time, and $\Gamma_f = \Gamma_f^D \cup \Gamma_f^N$. We assume that $|\Gamma_f^D| > 0$.

In the poroelastic region $\Omega_p$ we let $\Gamma_p = \Gamma_p^D \cup \Gamma_p^N = \tilde{\Gamma}_p^D \cup \tilde{\Gamma}_p^N$ and consider the quasi-static Biot system

$$-\nabla\cdot\boldsymbol{\sigma}_p = \mathbf{f}_p\,, \quad \mu(c)K^{-1}\mathbf{u}_p + \nabla p_p = \mathbf{0} \quad \text{in} \quad \Omega_p\times(0,T]\,, \tag{2.2a}$$

$$\frac{\partial}{\partial t}(s_0 p_p + \alpha\nabla\cdot\boldsymbol{\eta}_p) + \nabla\cdot\mathbf{u}_p = q_p \quad \text{in} \quad \Omega_p\times(0,T]\,, \tag{2.2b}$$

$$p_p = 0 \quad \text{on} \quad \Gamma_p^D\times(0,T]\,, \quad \mathbf{u}_p\cdot\mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p^N\times(0,T]\,, \tag{2.2c}$$

$$\boldsymbol{\eta}_p = \mathbf{0} \quad \text{on} \quad \tilde{\Gamma}_p^D\times(0,T]\,, \quad \boldsymbol{\sigma}_p\mathbf{n}_p = \mathbf{0} \quad \text{on} \quad \tilde{\Gamma}_p^N\times(0,T]\,, \tag{2.2d}$$

where the poroelastic stress tensor $\boldsymbol{\sigma}_p$ and the Cauchy stress tensor $\boldsymbol{\sigma}_e$ are defined, respectively, as

$$\boldsymbol{\sigma}_p = \boldsymbol{\sigma}_e - \alpha p_p\mathbf{I}\,, \quad \boldsymbol{\sigma}_e = \lambda_p(\nabla\cdot\boldsymbol{\eta}_p)\mathbf{I} + 2\mu_p\boldsymbol{\epsilon}(\boldsymbol{\eta}_p)\,,$$

where

$$0 < \lambda_{\min} \le \lambda_p(\mathbf{x}) \le \lambda_{\max}\,, \quad 0 < \mu_{\min} \le \mu_p(\mathbf{x}) \le \mu_{\max}\,, \tag{2.3}$$

are the Lamé parameters. In addition, $\alpha > 0$ is the Biot-Willis constant, $s_0 > 0$ is a storage coefficient and $K$ is a symmetric and uniformly positive definite rock permeability tensor, satisfying, for some constants $0 < k_{\min} \le k_{\max}$, $\forall\,\boldsymbol{\xi} \in \mathrm{R}^d$,

$$k_{\min}\,\boldsymbol{\xi}^{\mathrm{t}}\boldsymbol{\xi} \,\le\, \boldsymbol{\xi}^{\mathrm{t}}K(\mathbf{x})\boldsymbol{\xi} \,\le\, k_{\max}\boldsymbol{\xi}^{\mathrm{t}}\boldsymbol{\xi} \quad \forall\,\mathbf{x}\in\Omega_p\,. \tag{2.4}$$

We assume that $|\Gamma_p^D| > 0$ and $|\tilde{\Gamma}_p^D| > 0$. In addition, to simplify the characterization of the normal trace of the Darcy velocity on $\Gamma_{fp}$, we assume that $\mathrm{dist}(\Gamma_p^D, \Gamma_{fp}) \ge s > 0$. We assume that $\mu(\cdot)$ belongs to $C^1(\mathrm{R}^+ \cup \{0\})$ and there exist positive constants $\mu_L$, $\mu_U$, and $\hat{\mu}_U$, such that for any $x \in \mathrm{R}^+ \cup \{0\}$,

$$0 < \mu_L \le \mu(x) \le \mu_U \quad \text{and} \quad |\mu'(x)| \le \hat{\mu}_U\,. \tag{2.5}$$

We further assume that both $\mu(\cdot)$ and $\mu'(\cdot)$ are Lipschitz continuous, i.e., for all $x,y \in \mathrm{R}^+$, there exist positive constants $L_1$ and $L_2$ such that

$$|\mu(x) - \mu(y)| \le L_1\,|x-y| \quad \text{and} \quad |\mu'(x) - \mu'(y)| \le L_2\,|x-y|\,. \tag{2.6}$$

The coupling conditions on the fluid-poroelasticity interface $\Gamma_{fp} \times (0,T]$ are mass conservation, balance of force, balance of normal stress, and the Beavers-Joseph-Saffman (BJS) slip with friction condition:

$$\mathbf{u}_f \cdot \mathbf{n}_f + \left(\frac{\partial \boldsymbol{\eta}_p}{\partial t} + \mathbf{u}_p\right) \cdot \mathbf{n}_p = 0\,, \quad -(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f = p_p\,, \tag{2.7a}$$

$$\boldsymbol{\sigma}_f \mathbf{n}_f + \boldsymbol{\sigma}_p \mathbf{n}_p = \mathbf{0}\,, \quad -(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \boldsymbol{\tau}_{f,j} = \mu(c_0)\alpha_{BJS}\sqrt{K_j^{-1}}\left(\mathbf{u}_f - \frac{\partial \boldsymbol{\eta}_p}{\partial t}\right) \cdot \boldsymbol{\tau}_{f,j}\,, \tag{2.7b}$$

where $\tau_{f,j}$, $1 \leq j \leq d-1$, is an orthogonal system of the unit tangential vectors on $\Gamma_{fp}$, $K_j := (K\boldsymbol{\tau}_{f,j}) \cdot \boldsymbol{\tau}_{f,j}$, $\alpha_{BJS} \geq 0$ is an experimentally determined friction coefficient, and $\mu(c_0(\mathbf{x}))$ is the viscosity on the interface, where $c_0(\mathbf{x})$ is the initial concentration (see below (2.10)).

We set the initial condition as follows:

$$p_p(\mathbf{x},0) = p_{p,0}(\mathbf{x}) \quad \text{in} \quad \Omega_p\,.$$

The initial data of the remaining variables are constructed to satisfy a compatibility condition. The details are discussed in Theorem 3.2.

The Stokes–Biot model is coupled with the following transport equation for the solute concentration $c(\mathbf{x},t)$ in $\Omega$:

$$\phi\frac{\partial c}{\partial t} + \nabla \cdot (c\mathbf{u} - \mathbf{D}(\mathbf{u})\nabla c) = q\tilde{c} \quad \text{in} \quad \Omega \times (0,T]\,, \tag{2.8a}$$

$$c = 0 \quad \text{on} \quad \Gamma_{in} \times (0,T]\,, \quad (c\mathbf{u} - \mathbf{D}(\mathbf{u})\nabla c) \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_{out} \times (0,T]\,, \tag{2.8b}$$

where $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$ and $\partial\Omega = \Gamma_{in} \cup \Gamma_{out}$. We assume that $|\Gamma_{in}| > 0$. The choice of boundary conditions is made for simplicity. More general boundary conditions can also be handled. In addition, $0 < \phi_* \leq \phi(\mathbf{x}) \leq \phi^* \leq 1$ is the porosity of the medium in $\Omega_p$, while in $\Omega_f$ it is set to be 1, $\mathbf{u}$ is the velocity field over the entire domain $\Omega$, defined as $\mathbf{u}|_{\Omega_f} = \mathbf{u}_f$, $\mathbf{u}|_{\Omega_p} = \mathbf{u}_p$, $q$ is the source term given by $q|_{\Omega_f} = q_f$, $q|_{\Omega_p} = q_p$, respectively, whereas $\tilde{c}$ is defined as follows:

$$\tilde{c} = \begin{cases} \text{injected concentration } c_w, & \text{for } q > 0\,, \\ \text{resident concentration } c, & \text{for } q < 0\,. \end{cases}$$

Letting $q^+ = \max\{q, 0\}$ and $q^- = \min\{q, 0\}$ results in $q\tilde{c} = q^+ c_w + q^- c$. The diffusion-dispersion tensor $\mathbf{D}(\mathbf{u})$ in (2.8) is a non-linear function of the velocity that combines the effects of molecular diffusion and mechanical dispersion, defined as:

$$\mathbf{D}(\mathbf{u}) = \begin{cases} \phi\, d_m\, \mathbf{I} & \text{in } \Omega_f\,, \\ \phi\, d_m\, \mathbf{I} + |\mathbf{u}|\, [\alpha_l \mathbf{E}(\mathbf{u}) + \alpha_t\, (\mathbf{I} - \mathbf{E}(\mathbf{u}))] & \text{in } \Omega_p\,, \end{cases} \tag{2.9}$$

where $d_m$ is the molecular diffusivity, the constants $\alpha_l > 0$ and $\alpha_t > 0$ are the longitudinal and transverse dispersion, respectively, and $\mathbf{E}(\mathbf{u})$ is the projection tensor onto the direction of $\mathbf{u}$ with entries $(\mathbf{E}(\mathbf{u}))_{i,j} = \frac{\mathbf{u}_i \mathbf{u}_j}{|\mathbf{u}|^2}$. Here, for a vector $\boldsymbol{\zeta} \in \mathrm{R}^d$, $|\boldsymbol{\zeta}|^2 = \boldsymbol{\zeta}^{\mathrm{t}}\boldsymbol{\zeta}$. The transport model is complemented by the initial condition for the concentration,

$$c(\mathbf{x},0) = c_0(\mathbf{x}) \quad \text{in} \quad \Omega\,. \tag{2.10}$$

**Remark 2.1** *The choice of homogeneous inflow condition $c = 0$ on $\Gamma_{in}$ is made merely for simplicity of the presentation. The analysis can be easily extended to the case $c = c_{in}$ on $\Gamma_{in}$.*

### 2.2 Weak formulation

Let $(\cdot,\cdot)_{\Omega_\star}$, $\star \in \{f,p\}$, be the $L^2(\Omega_\star)$ inner product for scalar or vector valued functions. We use $(\cdot,\cdot)$ to denote the $L^2(\Omega)$ inner product. Let $\langle\cdot,\cdot\rangle_{\Gamma_{fp}}$ be the $L^2(\Gamma_{fp})$ inner product or duality pairing. We will use the standard notation for Sobolev spaces [27]. For a scalar space $U$, the space over the vector fields with components in $U$ is denoted by $\mathbf{U}$. We will use $C$ to denote a generic positive constant independent of the discretization parameter.

We start by introducing the functional spaces for the Stokes region $\Omega_f$, the Biot region $\Omega_p$ and the transport region $\Omega = \Omega_f \cup \Omega_p$:

$$
\begin{aligned}
\mathbf{V}_f &:= \Big\{\mathbf{v}_f \in \mathbf{H}^1(\Omega_f): \ \mathbf{v}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^D\Big\}, \qquad && W_f := L^2(\Omega_f)\,,\\
\mathbf{V}_p &:= \Big\{\mathbf{v}_p \in \mathbf{H}(\mathrm{div};\Omega_p): \ \mathbf{v}_p\cdot\mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p^N\Big\}, && W_p := L^2(\Omega_p)\,,\\
\mathbf{X}_p &:= \Big\{\boldsymbol{\xi}_p \in \mathbf{H}^1(\Omega_p): \ \boldsymbol{\xi}_p = \mathbf{0} \quad \text{on} \quad \tilde{\Gamma}_p^D\Big\}, && X_c := \Big\{\psi \in H^1(\Omega): \ \psi = 0 \quad \text{on} \quad \Gamma_{in}\Big\}\,,
\end{aligned}
$$

where $\mathbf{H}(\mathrm{div};\Omega_p) := \Big\{\mathbf{v}\in\mathbf{L}^2(\Omega_p): \quad \nabla\cdot\mathbf{v}\in L^2(\Omega_p)\Big\}$ endowed with the norm

$$
\|\mathbf{v}\|^2_{\mathbf{H}(\mathrm{div};\Omega_p)} := \|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega_p)} + \|\nabla\cdot\mathbf{v}\|^2_{L^2(\Omega_p)}\,.
$$

The above spaces are equipped with the norms

$$
\begin{aligned}
\|\mathbf{v}_f\|_{\mathbf{V}_f} &:= \|\mathbf{v}_f\|_{\mathbf{H}^1(\Omega_f)}\,, \qquad && \|w_f\|_{W_f} := \|w_f\|_{L^2(\Omega_f)}\,,\\
\|\mathbf{v}_p\|_{\mathbf{V}_p} &:= \|\mathbf{v}_p\|_{\mathbf{H}(\mathrm{div};\Omega_\mathrm{p})}\,, && \|w_p\|_{W_p} := \|w_p\|_{L^2(\Omega_p)}\,,\\
\|\boldsymbol{\xi}_p\|_{\mathbf{X}_p} &:= \|\boldsymbol{\xi}_p\|_{\mathbf{H}^1(\Omega_p)}\,, && \|\psi\|_{X_c} := \|\psi\|_{H^1(\Omega)}\,.
\end{aligned}
$$

Let $\partial_t := \frac{\partial}{\partial t}$. We introduce the Bochner spaces equipped with norms:

$$
\|\varphi\|^2_{L^2(0,T;X)} := \int_0^T \|\varphi(t)\|^2_X\,dt\,, \quad \|\varphi\|^2_{H^1(0,T;X)} := \int_0^T \Big(\|\varphi(t)\|^2_X + \|\partial_t\varphi(t)\|^2_X\Big)\,dt\,,
$$

$$
\|\varphi\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{t\in[0,T]}\|\varphi(t)\|_X\,, \quad \|\varphi\|_{W^{1,\infty}(0,T;X)} := \operatorname*{ess\,sup}_{t\in[0,T]}\Big\{\|\varphi(t)\|_X + \|\partial_t\varphi(t)\|_X\Big\}\,.
$$

We derive the Stokes–Biot weak formulation by testing (2.1)–(2.2) with arbitrary $\mathbf{v}_f\in\mathbf{V}_f$, $w_f\in W_f$, $\boldsymbol{\xi}_p\in\mathbf{X}_p$, $\mathbf{v}_p\in\mathbf{V}_p$, and $w_p\in W_p$, and integrating by parts the terms involving $\nabla\cdot\boldsymbol{\sigma}_f$, $\nabla\cdot\boldsymbol{\sigma}_p$, and $\nabla p_p$. This results in the following forms related to the Stokes, Darcy, and elasticity operators:

$$
a_f(\mathbf{u}_f,\mathbf{v}_f;c) := (2\mu(c)\boldsymbol{\epsilon}(\mathbf{u}_f),\boldsymbol{\epsilon}(\mathbf{v}_f))_{\Omega_f}, \quad a_p^d(\mathbf{u}_p,\mathbf{v}_p;c) := (\mu(c)K^{-1}\mathbf{u}_p,\mathbf{v}_p)_{\Omega_p},
$$

$$
a_p^e(\boldsymbol{\eta}_p,\boldsymbol{\xi}_p) := (2\mu_p\boldsymbol{\epsilon}(\boldsymbol{\eta}_p),\boldsymbol{\epsilon}(\boldsymbol{\xi}_p))_{\Omega_p} + (\lambda_p\nabla\cdot\boldsymbol{\eta}_p,\nabla\cdot\boldsymbol{\xi}_p)_{\Omega_p},
$$

$$
b_\star(\mathbf{v}_\star,w_\star) := -(\nabla\cdot\mathbf{v}_\star,w_\star)_{\Omega_\star},\ \star\in\{f,p\}, \quad b_e(\boldsymbol{\xi}_p,w_p) := -(\nabla\cdot\boldsymbol{\xi}_p,w_p)_{\Omega_p},
$$

and the interface term:

$$
I_{\Gamma_{fp}} := -\langle\boldsymbol{\sigma}_f\mathbf{n}_f,\mathbf{v}_f\rangle_{\Gamma_{fp}} - \big\langle\boldsymbol{\sigma}_p\mathbf{n}_p,\boldsymbol{\xi}_p\big\rangle_{\Gamma_{fp}} + \langle p_p,\mathbf{v}_p\cdot\mathbf{n}_p\rangle_{\Gamma_{fp}}\,.
$$

Introducing the Lagrange multiplier

$$
\lambda = -(\boldsymbol{\sigma}_f\mathbf{n}_f)\cdot\mathbf{n}_f = p_p \quad \text{on} \quad \Gamma_{fp}\,,
$$

and using the second equation in (2.7a) and (2.7b), we obtain

$$I_{\Gamma_{fp}} = a_{BJS}(\mathbf{u}_f, \partial_t \boldsymbol{\eta}_p; \mathbf{v}_f, \boldsymbol{\xi}_p) + b_\Gamma(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \lambda),$$

where

$$a_{BJS}(\mathbf{u}_f, \boldsymbol{\eta}_p; \mathbf{v}_f, \boldsymbol{\xi}_p) := \alpha_{BJS} \sum_{j=1}^{d-1} \left\langle \mu(c_0)\sqrt{K_j^{-1}}(\mathbf{u}_f - \boldsymbol{\eta}_p)\cdot \boldsymbol{\tau}_{f,j}, (\mathbf{v}_f - \boldsymbol{\xi}_p)\cdot \boldsymbol{\tau}_{f,j} \right\rangle_{\Gamma_{fp}},$$

$$b_\Gamma(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \lambda) := \left\langle \mathbf{v}_f \cdot \mathbf{n}_f + (\boldsymbol{\xi}_p + \mathbf{v}_p)\cdot \mathbf{n}_p, \lambda \right\rangle_{\Gamma_{fp}}.$$

We observe that, for any $\mathbf{v}_p \in \mathbf{V}_p \subset \mathbf{H}(\mathrm{div};\Omega_p)$, $\mathbf{v}_p \cdot \mathbf{n}_p|_{\Gamma_{fp}}$ can be identified as a functional in $H^{-1/2}(\Gamma_{fp})$ when $\mathrm{dist}(\Gamma_p^D, \Gamma_{fp}) \geq s > 0$ (see [3] for details). Therefore we take $\lambda \in \Lambda := H^{1/2}(\Gamma_{fp})$ with a norm $\|\cdot\|_\Lambda := \|\cdot\|_{H^{1/2}(\Gamma_{fp})}$, which guarantees that $b_\Gamma$ is continuous.

The continuity of normal flux condition, cf. the first equation in (2.7a), is imposed weakly by testing with a function $\nu \in \Lambda$. Finally, the transport equation (2.8) is tested with an arbitrary function $\psi \in X_c$. We thus arrive at the following weak formulation of the coupled Stokes–Biot–transport problem.

(**P**) Given $\mathbf{f}_f$, $\mathbf{f}_p$, $q_p$, $q_f$, $q$, and $c_w$, find $(\mathbf{u}_f, p_f, \boldsymbol{\eta}_p, \mathbf{u}_p, p_p, \lambda, c) : [0,T] \to \mathbf{V}_f \times W_f \times \mathbf{X}_p \times \mathbf{V}_p \times W_p \times \Lambda \times X_c$, such that for a.e. $t \in (0,T]$ and for all $\mathbf{v}_f \in \mathbf{V}_f$, $w_f \in W_f$, $\boldsymbol{\xi}_p \in \mathbf{X}_p$, $\mathbf{v}_p \in \mathbf{V}_p$, $w_p \in W_p$, $\nu \in \Lambda$, and $\psi \in X_c$,

$$a_f(\mathbf{u}_f, \mathbf{v}_f; c) + b_f(\mathbf{v}_f, p_f) + a_{BJS}(\mathbf{u}_f, \partial_t\boldsymbol{\eta}_p; \mathbf{v}_f, 0) + b_\Gamma(\mathbf{v}_f, 0, 0; \lambda) = (\mathbf{f}_f, \mathbf{v}_f)_{\Omega_f}, \tag{2.11a}$$

$$-\, b_f(\mathbf{u}_f, w_f) = (q_f, w_f)_{\Omega_f}, \tag{2.11b}$$

$$a_p^e(\boldsymbol{\eta}_p, \boldsymbol{\xi}_p) + \alpha b_e(\boldsymbol{\xi}_p, p_p) + a_{BJS}(\mathbf{u}_f, \partial_t\boldsymbol{\eta}_p; 0, \boldsymbol{\xi}_p) + b_\Gamma(0,0,\boldsymbol{\xi}_p;\lambda) = (\mathbf{f}_p, \boldsymbol{\xi}_p)_{\Omega_p}, \tag{2.11c}$$

$$a_p^d(\mathbf{u}_p, \mathbf{v}_p; c) + b_p(\mathbf{v}_p, p_p) + b_\Gamma(0, \mathbf{v}_p, 0; \lambda) = 0, \tag{2.11d}$$

$$s_0(\partial_t p_p, w_p)_{\Omega_p} - \alpha b_e(\partial_t \boldsymbol{\eta}_p, w_p) - b_p(\mathbf{u}_p, w_p) = (q_p, w_p)_{\Omega_p}, \tag{2.11e}$$

$$b_\Gamma(\mathbf{u}_f, \mathbf{u}_p, \partial_t\boldsymbol{\eta}_p; \nu) = 0, \tag{2.11f}$$

$$(\phi\partial_t c, \psi) + (\mathbf{D}(\mathbf{u})\nabla c, \nabla\psi) - (c\mathbf{u}, \nabla\psi) - (q^- c, \psi) = (q^+ c_w, \psi). \tag{2.11g}$$

We next define the following seminorm on the interface $\Gamma_{fp}$ for $\mathbf{v}_f \in \mathbf{V}_f$ and $\boldsymbol{\xi}_p \in \mathbf{X}_p$:

$$|\mathbf{v}_f - \boldsymbol{\xi}_p|^2_{BJS} = \sum_{j=1}^{d-1} \left\langle (\mathbf{v}_f - \boldsymbol{\xi}_p)\cdot\boldsymbol{\tau}_{f,j}, (\mathbf{v}_f - \boldsymbol{\xi}_p)\cdot\boldsymbol{\tau}_{f,j} \right\rangle_{\Gamma_{fp}},$$

and the following product norms:

$$\|(\mathbf{v}_f, \mathbf{v}_p)\|^2_{\mathbf{V}_f\times\mathbf{V}_p} := \|\mathbf{v}_f\|^2_{\mathbf{V}_f} + \|\mathbf{v}_p\|^2_{\mathbf{V}_p}, \quad \|(w_f, w_p, \nu)\|^2_{W_f\times W_p\times\Lambda} := \|w_f\|^2_{W_f} + \|w_p\|^2_{W_p} + \|\nu\|^2_\Lambda.$$

We now establish stability properties for $a_f$, $a_p^d$, $a_p^e$, and $a_{BJS}$, the inf-sup conditions for $b_f$, $b_p$, and $b_\Gamma$, as well as a useful result that will be applied later.

**Lemma 2.1** *Given $c \in X_c$. Then, there exist positive constants $c_f, c^f, c_p, c^p, c_e, c^e, c_I$, and $c^I$ such that for all $\mathbf{u}_f$, $\mathbf{v}_f \in \mathbf{V}_f$, $\mathbf{u}_p$, $\mathbf{v}_p \in \mathbf{V}_p$, $\boldsymbol{\eta}_p$, $\boldsymbol{\xi}_p \in \mathbf{X}_p$, there hold*

$$c_f\|\mathbf{v}_f\|^2_{\mathbf{H}^1(\Omega_f)} \leq a_f(\mathbf{v}_f, \mathbf{v}_f; c), \qquad a_f(\mathbf{u}_f, \mathbf{v}_f; c) \leq c^f\|\mathbf{u}_f\|_{\mathbf{H}^1(\Omega_f)}\|\mathbf{v}_f\|_{\mathbf{H}^1(\Omega_f)},$$

$$c_p\|\mathbf{v}_p\|^2_{\mathbf{L}^2(\Omega_p)} \le a^d_p(\mathbf{v}_p,\mathbf{v}_p;c)\,, \qquad a^d_p(\mathbf{u}_p,\mathbf{v}_p;c) \le c^p\|\mathbf{u}_p\|_{\mathbf{L}^2(\Omega_p)}\|\mathbf{v}_p\|_{\mathbf{L}^2(\Omega_p)}\,,$$
$$c_e\|\boldsymbol{\xi}_p\|^2_{\mathbf{H}^1(\Omega_p)} \le a^e_p(\boldsymbol{\xi}_p,\boldsymbol{\xi}_p), \qquad a^e_p(\boldsymbol{\eta}_p,\boldsymbol{\xi}_p) \le c^e\|\boldsymbol{\eta}_p\|_{\mathbf{H}^1(\Omega_p)}\|\boldsymbol{\xi}_p\|_{\mathbf{H}^1(\Omega_p)}\,,$$
$$c_I|\mathbf{v}_f-\boldsymbol{\xi}_p|^2_{BJS} \le a_{BJS}(\mathbf{v}_f,\boldsymbol{\xi}_p;\mathbf{v}_f,\boldsymbol{\xi}_p), \quad a_{BJS}(\mathbf{u}_f,\boldsymbol{\eta}_p;\mathbf{v}_f,\boldsymbol{\xi}_p) \le c^I|\mathbf{u}_f-\boldsymbol{\eta}_p|_{BJS}\,|\mathbf{v}_f-\boldsymbol{\xi}_p|_{BJS}\,.$$

*Proof.* We first recall Korn's inequality: given $\star \in \{f,p\}$, there exists a constant $K_\star > 0$ such that $\|\boldsymbol{\epsilon}(\mathbf{v})\|_{\mathbf{L}^2(\Omega_\star)} \ge K_\star\|\mathbf{v}\|_{\mathbf{H}^1(\Omega_\star)}, \quad \forall\,\mathbf{v}\in\{\mathbf{V}_f,\mathbf{X}_p\}$. Then, using (2.3), (2.4), and (2.5), we obtain the results with constants $c_f = 2\mu_L K_f^2$, $c^f = 2\mu_U$, $c_p = \mu_L/k_{\max}$, $c^p := \mu_U/k_{\min}$, $c_e = 2\mu_{\min}K_p^2$, $c^e = 2\mu_{\max}+\lambda_{\max}$, $c_I = \alpha_{BJS}\mu_L/\sqrt{k_{\max}}$, and $c^I = \alpha_{BJS}\mu_U/\sqrt{k_{\min}}$. □

**Lemma 2.2** *There exists a constant $\beta_1 > 0$ such that for all $(w,\nu)\in W\times\Lambda$, there holds*

$$\sup_{\mathbf{0}\neq(\mathbf{v}_f,\mathbf{v}_p)\in\mathbf{V}_f\times\mathbf{V}_p}\frac{b_f(\mathbf{v}_f,w_f)+b_p(\mathbf{v}_p,w_p)+b_\Gamma(\mathbf{v}_f,\mathbf{v}_p,0;\nu)}{\|(\mathbf{v}_f,\mathbf{v}_p)\|_{\mathbf{V}_f\times\mathbf{V}_p}} \ge \beta_1\,\|(w_f,w_p,\nu)\|_{W_f\times W_p\times\Lambda}\,. \tag{2.12}$$

*Proof.* This follows from a slight adaptation of the arguments used in the proofs of [33, Lemmas 3.1 and 3.2]. □

**Lemma 2.3** *Let $\mathbf{u}_p\in L^\infty(0,T;\mathbf{L}^\infty(\Omega_p))$. It holds that for any $\boldsymbol{\zeta}\in \mathrm{R}^d$,*

$$\forall\,(\mathbf{x},t)\in\Omega_p\times(0,T]\,,\quad \phi_* d_m|\boldsymbol{\zeta}|^2 \,\le\, \mathbf{D}(\mathbf{u})\boldsymbol{\zeta}\cdot\boldsymbol{\zeta} \,\le\, (\phi^* d_m + (\alpha_l+\alpha_t)|\mathbf{u}|)|\boldsymbol{\zeta}|^2\,. \tag{2.13}$$

*Assuming that $\mathbf{u}_p\in W^{1,\infty}(0,T;\mathbf{L}^\infty(\Omega_p))$, it holds that for any $\boldsymbol{\zeta},\boldsymbol{\chi}\in\mathrm{R}^d$,*

$$\forall\,(\mathbf{x},t)\in\Omega_p\times(0,T],\quad \partial_t\mathbf{D}(\mathbf{u})\boldsymbol{\zeta}\cdot\boldsymbol{\chi} \,\le\, 5(\alpha_t+\alpha_l)|\partial_t\mathbf{u}|\,|\boldsymbol{\zeta}\cdot\boldsymbol{\chi}|\,. \tag{2.14}$$

*If $\mathbf{u}^1_p,\mathbf{u}^2_p\in L^\infty(0,T;\mathbf{L}^\infty(\Omega_p))$, then it holds that for $1\le i,j\le d$,*

$$\forall\,(\mathbf{x},t)\in\Omega_p\times(0,T]\,,\quad \left|(\mathbf{D}(\mathbf{u}^1))_{i,j}-(\mathbf{D}(\mathbf{u}^2))_{i,j}\right| \le 3(\alpha_l+\alpha_t)|\mathbf{u}^1-\mathbf{u}^2|\,. \tag{2.15}$$

*Proof.* The inequalities in (2.13) follow from the definition of $\mathbf{D}$ (cf. (2.9)) and the fact that the eigenvalues of $\mathbf{E}(\mathbf{u})$ are 1 and 0. To obtain (2.14), we observe that in $\Omega_p$, there holds

$$\partial_t\mathbf{D}(\mathbf{u}) = \partial_t|\mathbf{u}|\,[\alpha_l\mathbf{E}(\mathbf{u})+\alpha_t(\mathbf{I}-\mathbf{E}(\mathbf{u}))] + (\alpha_l-\alpha_t)|\mathbf{u}|\partial_t\mathbf{E}(\mathbf{u})\,. \tag{2.16}$$

Next, using the fact that $\partial_t|\mathbf{u}|\le|\partial_t\mathbf{u}|$, we easily deduce that

$$\partial_t|\mathbf{u}|\,[\alpha_l\mathbf{E}(\mathbf{u})+\alpha_t(\mathbf{I}-\mathbf{E}(\mathbf{u}))]\,\boldsymbol{\zeta}\cdot\boldsymbol{\chi} \le (\alpha_t+\alpha_l)|\partial_t\mathbf{u}|\,|\boldsymbol{\zeta}\cdot\boldsymbol{\chi}|\,. \tag{2.17}$$

In turn, the identity

$$\partial_t(\mathbf{E}(\mathbf{u})) = \partial_t\left(\frac{\mathbf{u}\mathbf{u}^T}{|\mathbf{u}|^2}\right) = \frac{(\partial_t\mathbf{u})\mathbf{u}^T+\mathbf{u}(\partial_t\mathbf{u})^T}{|\mathbf{u}|^2} - \frac{2(\mathbf{u}\cdot\partial_t\mathbf{u})}{|\mathbf{u}|^2}\mathbf{E}(\mathbf{u})\,,$$

implies that

$$(\alpha_l-\alpha_t)|\mathbf{u}|\partial_t\mathbf{E}(\mathbf{u})\boldsymbol{\zeta}\cdot\boldsymbol{\chi} \le 4(\alpha_t+\alpha_l)|\partial_t\mathbf{u}|\,|\boldsymbol{\zeta}\cdot\boldsymbol{\chi}|\,. \tag{2.18}$$

Thus, combining (2.16) with (2.17)–(2.18), we obtain (2.14). Finally, (2.15) follows from the following property for the $(i,j)$-entry of $|\mathbf{u}^1|\mathbf{E}(\mathbf{u}^1)-|\mathbf{u}^2|\mathbf{E}(\mathbf{u}^2)$, which is established in [35, Theorem 3.1]:

$$\left|\frac{\mathbf{u}^1_i\mathbf{u}^1_j}{|\mathbf{u}^1|}-\frac{\mathbf{u}^2_i\mathbf{u}^2_j}{|\mathbf{u}^2|}\right| \le 3|\mathbf{u}^1-\mathbf{u}^2|\,.$$

□

The analysis of (**P**) is challenging due to its fully coupled nature and nonlinearities. To address these difficulties, we proceed with the analysis in the following steps:

- Consider two linearized and decoupled subproblems, denoted by (**LP1**) and (**LP2**).
- Study their semi-discrete Galerkin approximations, (**LGP1**) and (**LGP2**), and establish well-posedness using the theories of differential-algebraic equations (DAEs) and ordinary differential equations (ODEs), respectively (Section 3.2).
- Derive stability bounds for (**LGP1**) and (**LGP2**) (Section 3.3) and pass to the limit to obtain the well-posedness of (**LP1**) and (**LP2**) (Section 3.4).
- Finally, use a fixed-point iteration argument to establish the well-posedness of (**P**) (Section 4).

# 3 Linearized and decoupled subproblems

In this section, we introduce and analyze a linearized weak formulation related to (**P**). To this end, we first replace the terms $a_f(\mathbf{u}_f, \mathbf{v}_f; c)$ in (2.11a) and $a_p^d(\mathbf{u}_p, \mathbf{v}_p; c)$ in (2.11d) with $a_f(\mathbf{u}_f, \mathbf{v}_f; \gamma)$ and $a_p^d(\mathbf{u}_p, \mathbf{v}_p; \gamma)$, respectively, where $\gamma \in L^\infty(0, T; L^\infty(\Omega))$ is given. Next, in the transport equation (2.11g), we replace the terms $(c\mathbf{u}, \nabla\psi)$ and $(\mathbf{D}(\mathbf{u})\nabla c, \nabla\psi)$ with $(c\boldsymbol{\theta}, \nabla\psi)$ and $(\mathbf{D}(\boldsymbol{\theta})\nabla c, \nabla\psi)$, respectively, for some given $\boldsymbol{\theta} \in L^\infty(0, T; \mathbf{L}^\infty(\Omega))$. We then consider the following decoupled linearized problems.

(**LP1**) Given $\mathbf{f}_f$, $\mathbf{f}_p$, $q_p$, $q_f$, and $\gamma$, find $(\mathbf{u}_f, p_f, \boldsymbol{\eta}_p, \mathbf{u}_p, p_p, \lambda) : [0, T] \to \mathbf{V}_f \times W_f \times \mathbf{X}_p \times \mathbf{V}_p \times W_p \times \Lambda$, such that for a.e. $t \in (0, T]$ and for all $\mathbf{v}_f \in \mathbf{V}_f$, $w_f \in W_f$, $\boldsymbol{\xi}_p \in \mathbf{X}_p$, $\mathbf{v}_p \in \mathbf{V}_p$, $w_p \in W_p$, and $\nu \in \Lambda$,

$$a_f(\mathbf{u}_f, \mathbf{v}_f; \gamma) + b_f(\mathbf{v}_f, p_f) + a_{BJS}(\mathbf{u}_f, \partial_t\boldsymbol{\eta}_p; \mathbf{v}_f, 0) + b_\Gamma(\mathbf{v}_f, 0, 0; \lambda) = (\mathbf{f}_f, \mathbf{v}_f)_{\Omega_f}\,, \tag{3.1a}$$

$$- b_f(\mathbf{u}_f, w_f) = (q_f, w_f)_{\Omega_f}\,, \tag{3.1b}$$

$$a_p^e(\boldsymbol{\eta}_p, \boldsymbol{\xi}_p) + \alpha b_e(\boldsymbol{\xi}_p, p_p) + a_{BJS}(\mathbf{u}_f, \partial_t\boldsymbol{\eta}_p; 0, \boldsymbol{\xi}_p) + b_\Gamma(0, 0, \boldsymbol{\xi}_p; \lambda) = (\mathbf{f}_p, \boldsymbol{\xi}_p)_{\Omega_p}\,, \tag{3.1c}$$

$$a_p^d(\mathbf{u}_p, \mathbf{v}_p; \gamma) + b_p(\mathbf{v}_p, p_p) + b_\Gamma(0, \mathbf{v}_p, 0; \lambda) = 0\,, \tag{3.1d}$$

$$s_0(\partial_t p_p, w_p)_{\Omega_p} - \alpha b_e(\partial_t\boldsymbol{\eta}_p, w_p) - b_p(\mathbf{u}_p, w_p) = (q_p, w_p)_{\Omega_p}\,, \tag{3.1e}$$

$$b_\Gamma(\mathbf{u}_f, \mathbf{u}_p, \partial_t\boldsymbol{\eta}_p; \nu) = 0\,. \tag{3.1f}$$

(**LP2**) Given $q$, $c_w$, and $\boldsymbol{\theta}$, find $c : [0, T] \to X_c$, such that for a.e. $t \in (0, T]$ and for all $\psi \in X_c$,

$$(\phi\partial_t c, \psi) + (\mathbf{D}(\boldsymbol{\theta})\nabla c, \nabla\psi) - (c\boldsymbol{\theta}, \nabla\psi) - (q^- c, \psi) = (q^+ c_w, \psi)\,. \tag{3.2}$$

In order to prove the well-posedness of problems (**LP1**) and (**LP2**), and following a similar approach to [12], we introduce their semi-discrete Galerkin approximations by discretizing in space. We prove the solvability of (**LP1**) using the theory of differential-algebraic equations (DAE), while for (**LP2**), we apply the theory of ordinary differential equations (ODE). We then obtain stability estimates for the Galerkin solutions, use weak compactness, and pass to the limit to establish the existence and stability of solutions to (**LP1**) and (**LP2**).

## 3.1 Semi-discrete continuous-in-time Galerkin approximation

Let $\mathcal{T}_h = \mathcal{T}_h^f \cup \mathcal{T}_h^p$, where $\mathcal{T}_h^f$ and $\mathcal{T}_h^p$ are shape-regular partitions of $\Omega_f$ and $\Omega_p$, respectively, both consisting of affine elements with maximal element diameter $h$. The two partitions match at the interface $\Gamma_{fp}$. For the discretization of the fluid velocity and pressure we choose inf-sup stable finite element spaces $\mathbf{V}_{fh} \subset \mathbf{V}_f$ and $W_{fh} \subset W_f$, such as MINI elements or Taylor-Hood spaces. For the discretization of the Darcy equation, we choose $\mathbf{V}_{ph} \subset \mathbf{V}_p$ and $W_{ph} \subset W_p$ to be any inf-sup stable mixed finite element spaces, such as Raviart–Thomas or Brezzi–Douglas–Marini spaces [8]. In turn, let $\mathbf{X}_{ph} \subset \mathbf{X}_p$ and $X_{ch} \subset X_c$ be conforming Lagrangian finite element spaces. Finally, we take a conforming finite element space $\Lambda_h \subset \Lambda$, defined on $\mathcal{T}_h|_{\Gamma_{fp}}$, which is chosen as follows. If the normal trace of the space $\mathbf{V}_{ph}$ contains piecewise polynomials in $P_k$ on simplices or $Q_k$ on cubes with $k \geq 1$, we take $\Lambda_h$ to be the space of continuous piecewise polynomials in $P_k$ or $Q_k$ on the trace of the neighboring subdomain grids. Here $P_k$ denotes the polynomials of a total degree up to $k$ and $Q_k$ stands for polynomials of a degree up to $k$ in each variable. In the case $k = 0$, we take $\Lambda_h$ to be the space of continuous piecewise polynomials in $P_1$ or $Q_1$ on a grid obtained by coarsening by two the trace of the subdomain grids. This choice is made to ensure inf-sup stability for the interface bilinear form $b_\Gamma$, stated in the following lemma.

**Lemma 3.1** *There exists a constant $\beta_2 > 0$ such that for all $(w_{fh}, w_{ph}, \nu_h) \in W_{fh} \times W_{ph} \times \Lambda_h$, there holds*

$$\sup_{\mathbf{0} \neq (\mathbf{v}_{fh}, \mathbf{v}_{ph}) \in \mathbf{V}_f \times \mathbf{V}_p} \frac{b_f(\mathbf{v}_{fh}, w_{fh}) + b_p(\mathbf{v}_{ph}, w_{ph}) + b_\Gamma(\mathbf{v}_{fh}, \mathbf{v}_{ph}, 0; \nu_h)}{\|(\mathbf{v}_{fh}, \mathbf{v}_{ph})\|_{\mathbf{V}_f \times \mathbf{V}_p}} \geq \beta_2 \, \|(w_{fh}, w_{ph}, \nu_h)\|_{W_f \times W_p \times \Lambda} \, . \tag{3.3}$$

*Proof.* The result is proven in [36] (see also [33, Lemma 4.7] for $k \geq 1$) in the case of velocity boundary conditions on $\partial\Omega$ by restricting the mean value of $W_{ph} \times W_{fh}$. It can be easily verified that, since $|\Gamma_p^D| > 0$, the result holds with no restriction on $W_{ph} \times W_{fh}$. □

We consider the following linear and fully decoupled semi-discrete problems.

(**LGP1**) Given $\mathbf{f}_f$, $\mathbf{f}_p$, $q_p$, $q_f$, and $\gamma$, find $(\mathbf{u}_{fh}, p_{fh}, \boldsymbol{\eta}_{ph}, \mathbf{u}_{ph}, p_{ph}, \lambda_h) : [0,T] \to \mathbf{V}_{fh} \times W_{fh} \times \mathbf{X}_{ph} \times \mathbf{V}_{ph} \times W_{ph} \times \Lambda_h$, such that for a.e. $t \in (0,T]$ and for all $\mathbf{v}_{fh} \in \mathbf{V}_{fh}$, $w_{fh} \in W_{fh}$, $\boldsymbol{\xi}_{ph} \in \mathbf{X}_{ph}$, $\mathbf{v}_{ph} \in \mathbf{V}_{ph}$, $w_{ph} \in W_{ph}$, and $\nu_h \in \Lambda_h$,

$$a_f(\mathbf{u}_{fh}, \mathbf{v}_{fh}; \gamma) + b_f(\mathbf{v}_{fh}, p_{fh}) + a_{BJS}(\mathbf{u}_{fh}, \partial_t \boldsymbol{\eta}_{ph}; \mathbf{v}_{fh}, 0) + b_\Gamma(\mathbf{v}_{fh}, 0, 0; \lambda) = (\mathbf{f}_f, \mathbf{v}_{fh})_{\Omega_f} \, , \tag{3.4a}$$

$$- b_f(\mathbf{u}_{fh}, w_{fh}) = (q_f, w_{fh})_{\Omega_f} \, , \tag{3.4b}$$

$$a_p^e(\boldsymbol{\eta}_{ph}, \boldsymbol{\xi}_{ph}) + \alpha b_e(\boldsymbol{\xi}_{ph}, p_{ph}) + a_{BJS}(\mathbf{u}_{fh}, \partial_t \boldsymbol{\eta}_{ph}; 0, \boldsymbol{\xi}_{ph}) + b_\Gamma(0, 0, \boldsymbol{\xi}_{ph}; \lambda_h) = (\mathbf{f}_p, \boldsymbol{\xi}_{ph})_{\Omega_p} \, , \tag{3.4c}$$

$$a_p^d(\mathbf{u}_{ph}, \mathbf{v}_{ph}; \gamma) + b_p(\mathbf{v}_{ph}, p_{ph}) + b_\Gamma(0, \mathbf{v}_{ph}, 0; \lambda_h) = 0 \, , \tag{3.4d}$$

$$s_0(\partial_t p_{ph}, w_{ph})_{\Omega_p} - \alpha b_e(\partial_t \boldsymbol{\eta}_{ph}, w_{ph}) - b_p(\mathbf{u}_{ph}, w_{ph}) = (q_p, w_{ph})_{\Omega_p} \, , \tag{3.4e}$$

$$b_\Gamma(\mathbf{u}_{fh}, \mathbf{u}_{ph}, \partial_t \boldsymbol{\eta}_{ph}; \nu_h) = 0 \, . \tag{3.4f}$$

(**LGP2**) Given $q$, $c_w$, and $\boldsymbol{\theta}$, find $c_h : [0,T] \to X_{ch}$, such that for a.e. $t \in (0,T]$ and for all $\psi_h \in X_{ch}$, there holds

$$(\phi \partial_t c_h, \psi_h) + (\mathbf{D}(\boldsymbol{\theta}) \nabla c_h, \nabla \psi_h) - (c_h \boldsymbol{\theta}, \nabla \psi_h) - (q^- c_h, \psi_h) = (q^+ c_w, \psi_h) \, . \tag{3.5}$$

Let $\{\boldsymbol{\varphi}_{\mathbf{u}_f,i}\}$, $\{\boldsymbol{\varphi}_{\mathbf{u}_p,i}\}$, $\{\boldsymbol{\varphi}_{\boldsymbol{\eta}_p,i}\}$, $\{\varphi_{p_f,i}\}$, $\{\varphi_{p_p,i}\}$, $\{\varphi_{\lambda,i}\}$, and $\{\varphi_{c,i}\}$ be the basis of $\mathbf{V}_{fh}$, $\mathbf{V}_{ph}$, $\mathbf{X}_{ph}$, $W_{fh}$, $W_{ph}$, $\Lambda_h$, and $X_{ch}$, respectively. We introduce the following matrices:

$$(M_p)_{ij} = (\boldsymbol{\varphi}_{p_p,j}, \boldsymbol{\varphi}_{p_p,i})_{\Omega_p} \, ; \quad (M_c)_{ij} = (\phi \varphi_{c,j}, \varphi_{c,i}) \, ;$$

$$(A_f)_{ij} = a_f(\boldsymbol{\varphi}_{\mathbf{u}_f,j}, \boldsymbol{\varphi}_{\mathbf{u}_f,i}; \gamma) \, ; \quad (A_p^d)_{ij} = a_p^d(\boldsymbol{\varphi}_{\mathbf{u}_p,j}, \boldsymbol{\varphi}_{\mathbf{u}_p,i}; \gamma) \, ; \quad (A_p^e)_{i,j} = a_p^e(\boldsymbol{\varphi}_{\boldsymbol{\eta}_p,j}, \boldsymbol{\varphi}_{\boldsymbol{\eta}_p,i}) \, ;$$

$$(A_{BJS}^{ff})_{ij} = a_{BJS}(\boldsymbol{\varphi}_{\mathbf{u}_f,j}, 0; \boldsymbol{\varphi}_{\mathbf{u}_f,i}, 0) \, ; \quad (A_{BJS}^{fe})_{ij} = a_{BJS}(\boldsymbol{\varphi}_{\mathbf{u}_f,j}, 0; 0, \boldsymbol{\varphi}_{\boldsymbol{\eta}_p,i}) \, ;$$

$$(A^{ee}_{BJS})_{ij} = a_{BJS}(0, \boldsymbol{\varphi}_{\boldsymbol{\eta}_p,j}; 0, \boldsymbol{\varphi}_{\boldsymbol{\eta}_p,i})\,;$$

$$(B_f)_{ij} = b_f(\boldsymbol{\varphi}_{\mathbf{u}_f,j}, \varphi_{p_f,i})\,;\;\; (B_p)_{ij} = b_p(\boldsymbol{\varphi}_{\mathbf{u}_p,j}, \varphi_{p_p,i})\,;\;\; (B_e)_{ij} = b_e(\boldsymbol{\varphi}_{\boldsymbol{\eta}_p,j}, \varphi_{p_p,i})\,;$$

$$(B_{f,\Gamma})_{ij} = b_\Gamma(\boldsymbol{\varphi}_{\mathbf{u}_f,j}, 0, 0; \varphi_{\lambda,i})\,;\quad (B_{p,\Gamma})_{ij} = b_\Gamma(0, \boldsymbol{\varphi}_{\mathbf{u}_p,j}, 0; \varphi_{\lambda,i})\,;\quad (B_{e,\Gamma})_{ij} = b_\Gamma(0, 0, \boldsymbol{\varphi}_{\boldsymbol{\eta}_p,j}; \varphi_{\lambda,i})\,;$$

$$(D^1_c)_{i,j} = (\mathbf{D}(\boldsymbol{\theta})\nabla\varphi_{c,j}, \nabla\varphi_{c,i})\,;\;\; (D^2_c)_{i,j} = -(\boldsymbol{\theta}\varphi_{c,j}, \nabla\varphi_{c,i})\,;\;\; (D^-_c)_{i,j} = -(q^-\varphi_{c,j}, \varphi_{c,i})\,.$$

For the right-hand side data of (**LGP1**) and (**LGP2**), let

$$(\mathcal{F}_{\mathbf{f}_f})_i = (\mathbf{f}_f, \boldsymbol{\varphi}_{\mathbf{u}_f,i})_{\Omega_f}\,;\quad (\mathcal{F}_{\mathbf{f}_p})_i = (\mathbf{f}_p, \boldsymbol{\varphi}_{\boldsymbol{\eta}_p,i})_{\Omega_p}\,;\quad (\mathcal{F}_{q_f})_i = (q_f, \varphi_{p_f,i})_{\Omega_f}\,;$$

$$(\mathcal{F}_{q_p})_i = (q_p, \varphi_{p_p,i})_{\Omega_p}\,;\quad (\mathcal{F}_c)_i = (q^+ c_w, \varphi_{c,i})_\Omega\,.$$

Next, substituting in (3.4a)–(3.4f), we express the following functions as expansions in terms of basis functions with time-dependent coefficient vectors:

$$\mathbf{u}_{fh}(\mathbf{x},t) = \sum_i \mathbf{u}_{f,i}(t)\boldsymbol{\varphi}_{\mathbf{u}_f,i}\,,\quad \mathbf{u}_{ph}(\mathbf{x},t) = \sum_i \mathbf{u}_{p,i}(t)\boldsymbol{\varphi}_{\mathbf{u}_p,i}\,,\quad \boldsymbol{\eta}_{ph}(\mathbf{x},t) = \sum_i \boldsymbol{\eta}_{p,i}(t)\boldsymbol{\varphi}_{\boldsymbol{\eta}_p,i}\,,$$

$$p_{fh}(\mathbf{x},t) = \sum_i p_{f,i}(t)\varphi_{p_f,i}\,,\quad p_{ph}(\mathbf{x},t) = \sum_i p_{p,i}(t)\varphi_{p_p,i}\,,\quad \text{and}\quad \lambda_h(\mathbf{x},t) = \sum_i \lambda_i(t)\varphi_{\lambda,i}\,.$$

The (time-dependent) coefficient vectors are given by $\overline{\mathbf{u}}_f, \overline{\mathbf{u}}_p, \overline{\boldsymbol{\eta}}_p, \overline{p}_f, \overline{p}_p$, and $\overline{\lambda}$. This leads to the system:

$$A_f\overline{\mathbf{u}}_f + B^{\mathrm{t}}_f\overline{p}_f + A^{ff}_{BJS}\,\overline{\mathbf{u}}_f + A^{fe}_{BJS}\partial_t\overline{\boldsymbol{\eta}}_p + B^{\mathrm{t}}_{f,\Gamma}\overline{\lambda} = \mathcal{F}_{\mathbf{f}_f}\,, \tag{3.6a}$$

$$-B_f\overline{\mathbf{u}}_f = \mathcal{F}_{q_f}\,, \tag{3.6b}$$

$$A^e_p\overline{\boldsymbol{\eta}}_p + \alpha B^{\mathrm{t}}_e\overline{p}_p + (A^{fe}_{BJS})^{\mathrm{t}}\overline{\mathbf{u}}_f + A^{ee}_{BJS}\partial_t\overline{\boldsymbol{\eta}}_p + B^{\mathrm{t}}_{e,\Gamma}\overline{\lambda} = \mathcal{F}_{\mathbf{f}_p}\,, \tag{3.6c}$$

$$A^d_p\overline{\mathbf{u}}_p + B^{\mathrm{t}}_p\overline{p}_p + B^{\mathrm{t}}_{p,\Gamma}\overline{\lambda} = 0\,, \tag{3.6d}$$

$$s_0 M_p\partial_t\overline{p}_p - \alpha B_e\partial_t\overline{\boldsymbol{\eta}}_p - B_p\overline{\mathbf{u}}_p = \mathcal{F}_{q_p}\,, \tag{3.6e}$$

$$B_{f,\Gamma}\overline{\mathbf{u}}_f + B_{p,\Gamma}\overline{\mathbf{u}}_p + B_{e,\Gamma}\partial_t\overline{\boldsymbol{\eta}}_p = 0\,. \tag{3.6f}$$

This is a DAE system

$$\mathbf{G}\,\frac{d}{dt}X(t) + \mathbf{H}\,X(t) = R(t)\,, \tag{3.7}$$

where

$$X(t) = \begin{pmatrix} \overline{\mathbf{u}}_f(t) \\ \overline{\mathbf{u}}_p(t) \\ \overline{\boldsymbol{\eta}}_p(t) \\ \overline{p}_f(t) \\ \overline{p}_p(t) \\ \overline{\lambda}(t) \end{pmatrix},\quad R(t) = \begin{pmatrix} \mathcal{F}_{\mathbf{f}_f} \\ 0 \\ \mathcal{F}_{\mathbf{f}_p} \\ \mathcal{F}_{q_f} \\ \mathcal{F}_{q_p} \\ 0 \end{pmatrix},\quad \mathbf{G} = \begin{pmatrix} 0 & 0 & A^{fe}_{BJS} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A^{ee}_{BJS} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha B_e & 0 & s_0 M_p & 0 \\ 0 & 0 & -B_{e,\Gamma} & 0 & 0 & 0 \end{pmatrix}_{6\times 6},$$

$$\text{and}\quad \mathbf{H} = \begin{pmatrix} A_f + A^{ff}_{BJS} & 0 & 0 & (B_f)^{\mathrm{t}} & 0 & (B_{f,\Gamma})^{\mathrm{t}} \\ 0 & A^d_p & 0 & 0 & (B_p)^{\mathrm{t}} & (B_{p,\Gamma})^{\mathrm{t}} \\ (A^{fe}_{BJS})^{\mathrm{t}} & 0 & A^e_p & 0 & \alpha(B_e)^{\mathrm{t}} & (B_{e,\Gamma})^{\mathrm{t}} \\ -B_f & 0 & 0 & 0 & 0 & 0 \\ 0 & -B_p & 0 & 0 & 0 & 0 \\ -B_{f,\Gamma} & -B_{p,\Gamma} & 0 & 0 & 0 & 0 \end{pmatrix}_{6\times 6}.$$

Similarly, we take $c_h(\mathbf{x},t) = \sum_i c_i(t)\varphi_{c,i}$ in (3.5), where $\overline{c}$ is the (time-dependent) coefficient vector. Denoting $\tilde{D} := D^1_c + D^2_c + D^-_c$, we obtain the following ODE system:

$$M_c\,\frac{d}{dt}\overline{c}(t) + \tilde{D}\,\overline{c}(t) = \mathcal{F}_c(t)\,. \tag{3.8}$$

## 3.2 Well-posedness of the semi-discrete Galerkin problems (LGP1) and (LGP2)

We next discuss the solvability of the initial value problems associated with (3.7) and (3.8). Assuming that $c_0 \in X_c$, we take initial data

$$c_{h,0} = I_{ch} c_0 \,, \tag{3.9}$$

where $I_{ch} : X_c \to X_{ch}$ is the Scott-Zhang interpolant [50]. Initial data compatible with the DAE system (3.7) are constructed in the proof of the following theorem.

**Theorem 3.2** *Let $\mathbf{f}_f \in L^\infty(0,T;\mathbf{V}_f')$, $\mathbf{f}_p \in L^\infty(0,T;\mathbf{V}_p')$, $q_f \in L^\infty(0,T;W_f')$, $q_p \in L^\infty(0,T;W_p')$, and $\gamma \in L^\infty(0,T;L^\infty(\Omega))$. Assume that $c_0 \in X_c$ and $p_{p,0} \in H_p$, where*

$$H_p := \left\{ w_p \in H^1(\Omega_p) : \ \frac{1}{\mu(c_0)} K\nabla w_p \in \mathbf{H}^1(\Omega_p), \ \frac{1}{\mu(c_0)} K\nabla w_p = \mathbf{0} \ \text{on } \Gamma_p^N, \ w_p = 0 \ \text{on } \Gamma_p^D \right\}.$$

*Then there exists initial data $(\mathbf{u}_{fh,0}, p_{fh,0}, \boldsymbol{\eta}_{ph,0}, \mathbf{u}_{ph,0}, p_{ph,0}, \lambda_{h,0}) \in \mathbf{V}_{fh} \times W_{fh} \times \mathbf{X}_{ph} \times \mathbf{V}_{ph} \times W_{ph} \times \Lambda_h$ such that the Galerkin problem (**LGP1**), with initial conditions $p_{ph}(0) = p_{ph,0}$ and $\boldsymbol{\eta}_{ph}(0) = \boldsymbol{\eta}_{ph,0}$ has a unique solution. Moreover, the solution satisfies $\mathbf{u}_{fh}(0) = \mathbf{u}_{fh,0}$, $p_{fh}(0) = p_{fh,0}$, $\mathbf{u}_{ph}(0) = \mathbf{u}_{ph,0}$, and $\lambda_h(0) = \lambda_{h,0}$.*

*Let $q^- \in L^\infty(0,T;L^\infty(\Omega))$, $q^+ c_w \in L^\infty(0,T;X_c')$, $\boldsymbol{\theta} \in L^\infty(0,T;\mathbf{L}^\infty(\Omega))$, and $c_0 \in X_c$. Then the Galerkin problem (**LGP2**) with initial condition $c_h(0) = c_{h,0}$ has a unique solution.*

*Proof.* First, we note that the continuity of the forms established in Lemma 2.1, along with the assumed regularity of the data, guarantees that all terms in (**LGP1**) and (**LGP2**) are well-defined, resulting in a well-defined DAE system (3.7) and ODE system (3.8). Moreover, (3.7) is linear with constant coefficient matrices $\mathbf{G}$, $\mathbf{H}$, and a continuous $R(t)$. Similarly, (3.8) is linear with constant coefficient matrices $M_c$, $\tilde{D}$, and a continuous $\mathcal{F}_c(t)$.

We begin with proving the solvability of (**LGP1**). The proof is organized in three main steps: (1) prove the non-singularity of the matrix $\mathbf{G} + \mathbf{H}$ and utilize the DAE theory to establish existence of a solution; (2) construct compatible initial data for all variables; (3) derive an energy estimate to obtain uniqueness of the solution under a suitable initial condition.

**Existence for (LGP1).** Existence of a solution of the system (3.7) will be established using the DAE theory (see, for instance, [10, Theorem 2.3.1]). For the associated initial value problem, we need to construct initial data for all variables that are consistent with the DAE system. To address this issue, we consider an auxiliary DAE system by introducing a new variable $\mathbf{u}_{sh} \in \mathbf{X}_{ph}$ satisfying

$$(\mathbf{u}_{sh}, \mathbf{v}_{sh})_{\Omega_p} = (\partial_t \boldsymbol{\eta}_{ph}, \mathbf{v}_{sh})_{\Omega_p} \quad \forall\, \mathbf{v}_{sh} \in \mathbf{X}_{ph} \,, \tag{3.10}$$

and replacing $\partial_t \boldsymbol{\eta}_{ph}$ by $\mathbf{u}_{sh}$ in (3.4a), (3.4c), (3.4e), and (3.4f). Next, let $\mathbf{u}_{sh}(\mathbf{x},t) = \sum_i \mathbf{u}_{s,i}(t) \boldsymbol{\varphi}_{\boldsymbol{\eta}_p,i}$, and $\overline{\mathbf{u}}_s$ be the coefficient vector. This results in the extended DAE system

$$\tilde{\mathbf{G}} \frac{d}{dt} \tilde{X}(t) + \tilde{\mathbf{H}}\, \tilde{X}(t) = \tilde{R}(t) \,, \tag{3.11}$$

where $\tilde{X}(t)$, $\tilde{R}(t)$, $\tilde{\mathbf{G}}$, and $\tilde{\mathbf{H}}$ are defined, respectively, as follows:

$$\tilde{X}(t) = \begin{pmatrix} \overline{\mathbf{u}}_f(t) \\ \overline{\mathbf{u}}_p(t) \\ \overline{\boldsymbol{\eta}}_p(t) \\ \overline{\mathbf{u}}_s(t) \\ \overline{p}_f(t) \\ \overline{p}_p(t) \\ \overline{\lambda}(t) \end{pmatrix}, \quad \tilde{R}(t) = \begin{pmatrix} \mathcal{F}_{\mathbf{f}_f} \\ 0 \\ 0 \\ \mathcal{F}_{\mathbf{f}_p} \\ \mathcal{F}_{q_f} \\ \mathcal{F}_{q_p} \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{G}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_0 M_p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{7\times 7},$$

$$\text{and}\quad \tilde{\mathbf{H}} = \begin{pmatrix} A_f + A_{ff}^{BJS} & 0 & 0 & A_{fe}^{BJS} & (B_f)^{\mathrm{t}} & 0 & (B_{f,\Gamma})^{\mathrm{t}} \\ 0 & A_p^d & 0 & 0 & 0 & (B_p)^{\mathrm{t}} & (B_{p,\Gamma})^{\mathrm{t}} \\ 0 & 0 & 0 & -I & 0 & 0 & 0 \\ (A_{fe}^{BJS})^{\mathrm{t}} & 0 & A_p^e & A_{BJS}^{ee} & 0 & \alpha(B_e)^{\mathrm{t}} & (B_{e,\Gamma})^{\mathrm{t}} \\ -B_f & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -B_p & 0 & -\alpha B_e & 0 & 0 & 0 \\ -B_{f,\Gamma} & -B_{p,\Gamma} & 0 & -B_{e,\Gamma} & 0 & 0 & 0 \end{pmatrix}_{7\times 7}.$$

Note that any solution of (3.11) also solves (3.7). According to Theorem 2.3.1 in [10], (3.11) has a solution if the matrix $\omega\tilde{\mathbf{G}} + \tilde{\mathbf{H}}$ is non-singular for some $\omega \in \mathrm{R}$. We will prove that $\tilde{\mathbf{G}} + \tilde{\mathbf{H}}$ is non-singular by showing that the system $(\tilde{\mathbf{G}} + \tilde{\mathbf{H}})\tilde{X} = 0$ has only the zero solution. After eliminating $\overline{\mathbf{u}}_s$, this system results in $(\mathbf{G} + \mathbf{H})X = 0$, which implies

$$\begin{aligned} X^T(\mathbf{G} + \mathbf{H})X &= a_f(\mathbf{u}_{fh}, \mathbf{u}_{fh}; \gamma) + a_p^d(\mathbf{u}_{ph}, \mathbf{u}_{ph}; \gamma) + a_p^e(\boldsymbol{\eta}_{ph}, \boldsymbol{\eta}_{ph}) \\ &\quad + a_{BJS}(\mathbf{u}_{fh}, \boldsymbol{\eta}_{ph}; \mathbf{u}_{fh}, \boldsymbol{\eta}_{ph}) + s_0(p_{ph}, p_{ph})_{\Omega_p} = 0\,. \end{aligned}$$

Lemma 2.1 implies that $\mathbf{u}_{fh} = 0$, $\mathbf{u}_{ph} = 0$, $\boldsymbol{\eta}_{ph} = 0$, and $p_{ph} = 0$. The inf-sup condition (3.3) with $(w_{fh}, w_{ph}, \nu_h) = (p_{fh}, p_{ph}, \lambda_h)$, combined with (3.4a) and (3.4d), gives $p_{fh} = 0$ and $\lambda_h = 0$, while (3.10) implies that $\mathbf{u}_{sh} = 0$. We conclude that (3.11) has a solution.

**Construction of initial data for (LGP1).** To construct discrete initial data that are consistent with the DAE system (3.11), we first construct data for the continuous weak formulation (3.1). We then define the discrete data using elliptic projections.

Recall that $p_{p,0} \in H_p$ is given. Let $\mathbf{u}_{p,0} := -\frac{1}{\mu(c_0)} K\nabla p_{p,0}$. We have that $\mathbf{u}_{p,0} \in \mathbf{V}_p \cap \mathbf{H}^1(\Omega_p)$. Letting $\lambda_0 := p_{p,0}|_{\Gamma_{fp}} \in \Lambda$ and $c_0 \in X_c$, we conclude that

$$a_p^d(\mathbf{u}_{p,0}, \mathbf{v}_p; c_0) + b_p(\mathbf{v}_p, p_{p,0}) + \langle \mathbf{v}_p \cdot \mathbf{n}_p, \lambda_0 \rangle_{\Gamma_{fp}} = 0 \quad \forall\, \mathbf{v}_p \in \mathbf{V}_p\,. \tag{3.12}$$

Next, let $(\mathbf{u}_{f,0}, p_{f,0}) \in \mathbf{V}_f \times W_f$ be the solution to the Stokes problem

$$a_f(\mathbf{u}_{f,0}, \mathbf{v}_f; c_0) + b_f(\mathbf{v}_f, p_{f,0}) + \alpha_{BJS} \sum_{j=1}^{d-1} \left\langle \mu(c_0)\sqrt{K_j^{-1}}\mathbf{u}_{p,0} \cdot \boldsymbol{\tau}_{f,j}, \mathbf{v}_f \cdot \boldsymbol{\tau}_{f,j} \right\rangle_{\Gamma_{fp}}$$

$$+ \langle \mathbf{v}_f \cdot \mathbf{n}_f, \lambda_0 \rangle_{\Gamma_{fp}} = (\mathbf{f}_f(0), \mathbf{v}_f)_{\Omega_f} \quad \forall\, \mathbf{v}_f \in \mathbf{V}_f\,, \tag{3.13a}$$

$$- b_f(\mathbf{u}_{f,0}, w_f) = (q_f(0), w_f)_{\Omega_f} \quad \forall\, w_f \in W_f\,, \tag{3.13b}$$

where $\mathbf{u}_{p,0}$ and $\lambda_0$ are given as data. The well-posedness of (3.13) is ensured by Babuška–Brezzi theory (see, for instance, [37, Chapter 2]). Next, solve an elasticity problem for $\boldsymbol{\eta}_{p,0} \in \mathbf{X}_p$:

$$a_p^e(\boldsymbol{\eta}_{p,0}, \boldsymbol{\xi}_p) + \alpha b_e(\boldsymbol{\xi}_p, p_{p,0}) - \alpha_{BJS} \sum_{j=1}^{d-1} \left\langle \mu(c_0)\sqrt{K_j^{-1}}\mathbf{u}_{p,0} \cdot \boldsymbol{\tau}_{f,j}, \boldsymbol{\xi}_p \cdot \boldsymbol{\tau}_{f,j} \right\rangle_{\Gamma_{fp}}$$

$$+ \langle \boldsymbol{\xi}_p \cdot \mathbf{n}_p, \lambda_0 \rangle_{\Gamma_{fp}} = (\mathbf{f}_p(0), \boldsymbol{\xi}_p)_{\Omega_p} \quad \forall\, \boldsymbol{\xi}_p \in \mathbf{X}_p\,, \tag{3.14}$$

where $p_{p,0}$, $\mathbf{u}_{p,0}$, and $\lambda_0$ are given as data. The solvability of the problem (3.14) is ensured by Lax–Milgram Theorem [34]. Finally, let us define

$$\mathbf{u}_{s,0} := E_{\Omega_p}\Big(E_{0,\Gamma_p}(\mathbf{u}_{f,0}|_{\Gamma_{fp}}) - E_{0,\Gamma_p}(\mathbf{u}_{p,0}|_{\Gamma_{fp}})\Big) \in \mathbf{H}^1_{\Gamma_p}(\Omega_p) \subset \mathbf{X}_p\,,$$

where $E_{0,\Gamma_p}$ denotes the extension by zero to $\partial\Omega_p$, $E_{\Omega_p} : \mathbf{H}^{1/2}(\partial\Omega_p) \to \mathbf{H}^1(\Omega_p)$ is a continuous extension operator, and $\mathbf{H}^1_{\Gamma_p}(\Omega_p) := \big\{\mathbf{v}_p \in \mathbf{H}^1(\Omega_p) : \quad \mathbf{v}_p = \mathbf{0} \quad \text{on} \quad \Gamma_p\big\}$. Since $\mathbf{u}_{f,0} = \mathbf{0}$ on $\Gamma_f$ and $\mathbf{u}_{p,0} = \mathbf{0}$ on $\Gamma_p^N$, their extensions by zero are continuous in the $\mathbf{H}^{1/2}(\partial\Omega_p)$-norm and it holds that

$$
\begin{aligned}
\|\mathbf{u}_{s,0}\|_{\mathbf{H}^1(\Omega_p)} &\leq C\, \Big(\|E_{0,\Gamma_p}(\mathbf{u}_{f,0}|_{\Gamma_{fp}})\|_{\mathbf{H}^{1/2}(\partial\Omega_p)} + \|E_{0,\Gamma_p}(\mathbf{u}_{p,0}|_{\Gamma_{fp}})\|_{\mathbf{H}^{1/2}(\partial\Omega_p)}\Big) \\
&\leq C\, \Big(\|\mathbf{u}_{f,0}\|_{\mathbf{H}^{1/2}(\Gamma_{fp})} + \|\mathbf{u}_{p,0}\|_{\mathbf{H}^{1/2}(\Gamma_{fp})}\Big) \leq C\, \Big(\|\mathbf{u}_{f,0}\|_{\mathbf{H}^1(\Omega_f)} + \|\mathbf{u}_{p,0}\|_{\mathbf{H}^1(\Omega_p)}\Big), \quad (3.15)
\end{aligned}
$$

where we also used the trace inequality

$$\|\varphi\|_{H^{1/2}(\Gamma_{fp})} \leq C\|\varphi\|_{H^1(\Omega_\star)}, \quad \star \in \{f,p\}. \tag{3.16}$$

In addition, the definition of $\mathbf{u}_{s,0}$ implies that

$$b_\Gamma(\mathbf{u}_{f,0}, \mathbf{u}_{p,0}, \mathbf{u}_{s,0}; \nu) = 0 \quad \forall\, \nu \in \Lambda, \tag{3.17}$$

and enables us to replace $\mathbf{u}_{p,0} \cdot \boldsymbol{\tau}_{f,j}$ in the BJS terms in (3.13a) and (3.14) with $(\mathbf{u}_{f,0} - \mathbf{u}_{s,0}) \cdot \boldsymbol{\tau}_{f,j}$.

We proceed with the construction of the discrete initial data. Let $\mathbf{u}_{sh,0} := I_{sh}\mathbf{u}_{s,0}$, where $I_{sh} : \mathbf{H}^1(\Omega_p) \to \mathbf{X}_{ph}$ is the Scott–Zhang interpolant. Next, let $(\mathbf{u}_{fh,0}, p_{fh,0}, \mathbf{u}_{ph,0}, p_{ph,0}, \lambda_{h,0}) \in \mathbf{V}_{fh} \times W_{fh} \times \mathbf{V}_{ph} \times W_{ph} \times \Lambda_h$ be the solution to the Stokes–Darcy problem: for all $\mathbf{v}_{fh} \in \mathbf{V}_{fh}$, $w_{fh} \in W_{fh}$, $\mathbf{v}_{ph} \in \mathbf{V}_{ph}$, $w_{ph} \in W_{ph}$, and $\nu_h \in \Lambda_h$,

$$
\begin{aligned}
& a_f(\mathbf{u}_{fh,0}, \mathbf{v}_{fh}; c_0) + b_f(\mathbf{v}_{fh}, p_{fh,0}) + a_{BJS}(\mathbf{u}_{fh,0}, \mathbf{u}_{sh,0}; \mathbf{v}_{fh}, 0) + b_\Gamma(\mathbf{v}_{fh}, 0, 0; \lambda_{h,0}) \\
& \quad = a_f(\mathbf{u}_{f,0}, \mathbf{v}_{fh}; c_0) + b_f(\mathbf{v}_{fh}, p_{f,0}) + a_{BJS}(\mathbf{u}_{f,0}, \mathbf{u}_{s,0}; \mathbf{v}_{fh}, 0) + b_\Gamma(\mathbf{v}_{fh}, 0, 0; \lambda_0) \\
& \quad = (\mathbf{f}_f(0), \mathbf{v}_{fh})_{\Omega_f}, && (3.18a) \\
& -b_f(\mathbf{u}_{fh,0}, w_{fh}) = -b_f(\mathbf{u}_{f,0}, w_{fh}) = (q_f(0), w_{fh})_{\Omega_f}, && (3.18b) \\
& a_p^d(\mathbf{u}_{ph,0}, \mathbf{v}_{ph}; c_0) + b_p(\mathbf{v}_{ph}, p_{ph,0}) + b_\Gamma(0, \mathbf{v}_{ph}, 0; \lambda_{h,0}) \\
& \quad = a_p^d(\mathbf{u}_{p,0}, \mathbf{v}_{ph}; c_0) + b_p(\mathbf{v}_{ph}, p_{p,0}) + b_\Gamma(0, \mathbf{v}_{ph}, 0; \lambda_0) = 0, && (3.18c) \\
& -b_p(\mathbf{u}_{ph,0}, w_{ph}) = -b_p(\mathbf{u}_{p,0}, w_{ph}) = -(\nabla \cdot \mu(c_0)^{-1} K \nabla p_{p,0}, w_{ph})_{\Omega_p}, && (3.18d) \\
& b_\Gamma(\mathbf{u}_{fh,0}, \mathbf{u}_{ph,0}, \mathbf{u}_{sh,0}; \nu_h) = b_\Gamma(\mathbf{u}_{f,0}, \mathbf{u}_{p,0}, \mathbf{u}_{s,0}; \nu_h) = 0. && (3.18e)
\end{aligned}
$$

This is a well-posed Stokes–Darcy problem, due to the inf-sup condition (3.3), the arguments employed in [33, 42], and the Babuška–Brezzi theory. Finally, let $\boldsymbol{\eta}_{ph,0} \in \mathbf{X}_{ph}$ be the solution to the elasticity problem: for all $\boldsymbol{\xi}_{ph} \in \mathbf{X}_{ph}$,

$$
\begin{aligned}
& a_p^e(\boldsymbol{\eta}_{ph,0}, \boldsymbol{\xi}_{ph}) + \alpha b_e(\boldsymbol{\xi}_{ph}, p_{ph,0}) - a_{BJS}(\mathbf{u}_{fh,0}, \mathbf{u}_{sh,0}; 0, \boldsymbol{\xi}_{ph}) + b_\Gamma(0, 0, \boldsymbol{\xi}_{ph}; \lambda_{h,0}) \\
& = a_p^e(\boldsymbol{\eta}_{p,0}, \boldsymbol{\xi}_{ph}) + \alpha b_e(\boldsymbol{\xi}_{ph}, p_{p,0}) - a_{BJS}(\mathbf{u}_{f,0}, \mathbf{u}_{s,0}; 0, \boldsymbol{\xi}_{ph}) + b_\Gamma(0, 0, \boldsymbol{\xi}_{ph}; \lambda_0) = (\mathbf{f}_p(0), \boldsymbol{\xi}_{ph})_{\Omega_p}, \quad (3.19)
\end{aligned}
$$

where, we use $\mathbf{u}_{f,0}, \boldsymbol{\eta}_{p,0}, \mathbf{u}_{s,0}, p_{p,0}$, and $\lambda_0$ as data, whereas $\mathbf{u}_{sh,0}$ is the Scott–Zhang interpolant of $\mathbf{u}_{s,0}$, and $(\mathbf{u}_{fh,0}, p_{fh,0}, \mathbf{u}_{ph,0}, p_{ph,0}, \lambda_{h,0})$ is the solution of (3.18). Again, by the Lax–Milgram Theorem [34], we obtain the solvability of $\boldsymbol{\eta}_{ph,0}$ as the discrete initial condition. Based on (3.18)–(3.19), we have constructed initial data $(\mathbf{u}_{fh,0}, p_{fh,0}, \boldsymbol{\eta}_{ph,0}, \mathbf{u}_{ph,0}, p_{ph,0}, \lambda_{h,0}, \mathbf{u}_{sh,0})$ for the extended DAE system (3.11), which satisfy all equations in (3.11) at $t = 0$, except for the equations with time derivatives, cf. (3.6e) and (3.10). With the constructed initial data, these equations at $t = 0$ determine $\partial_t p_{ph}(0)$ and $\partial_t \boldsymbol{\eta}_{ph}(0)$. Since $\partial_t p_{ph}$, $\partial_t \boldsymbol{\eta}_{ph}$ appear only in (3.6e), (3.10), respectively, the two equations hold at $t = 0$, which does not lead to inconsistency with the rest of the equations.

Therefore there exists a solution to (3.11) satisfying $(\mathbf{u}_{fh}(0), p_{fh}(0), \boldsymbol{\eta}_{ph}(0), \mathbf{u}_{ph}(0), p_{ph}(0), \lambda_h(0), \mathbf{u}_{sh}(0)) = (\mathbf{u}_{fh,0}, p_{fh,0}, \boldsymbol{\eta}_{ph,0}, \mathbf{u}_{ph,0}, p_{ph,0}, \lambda_{h,0}, \mathbf{u}_{sh,0})$. Using (3.10) to eliminate $\mathbf{u}_{sh}$ from the system results in a solution to the original DAE system (3.7) satisfying $(\mathbf{u}_{fh}(0), p_{fh}(0), \boldsymbol{\eta}_{ph}(0), \mathbf{u}_{ph}(0), p_{ph}(0), \lambda_h(0)) = (\mathbf{u}_{fh,0}, p_{fh,0}, \boldsymbol{\eta}_{ph,0}, \mathbf{u}_{ph,0}, p_{ph,0}, \lambda_{h,0})$.

**Uniqueness of the solution to (LGP1).** Since (3.4a)–(3.4f) is linear, it is sufficient to prove that the problem with zero data has only the zero solution. Taking $(\mathbf{f}_f, q_f, \mathbf{f}_p, q_p) = (\mathbf{0}, 0, \mathbf{0}, 0)$ in (3.4a)–(3.4f), testing it with the solution $(\mathbf{u}_{fh}, p_{fh}, \partial_t\boldsymbol{\eta}_{ph}, \mathbf{u}_{ph}, p_{ph}, \lambda_h)$, and using Lemma 2.1, yields

$$s_0\|p_{ph}(t)\|^2_{L^2(\Omega_p)} + \|\boldsymbol{\eta}_{ph}(t)\|^2_{\mathbf{H}^1(\Omega_p)} + \int_0^t \Big(\|\mathbf{u}_{fh}\|^2_{\mathbf{H}^1(\Omega_f)} + \|\mathbf{u}_{ph}\|^2_{\mathbf{L}^2(\Omega_p)} + |\mathbf{u}_{fh} - \partial_t\boldsymbol{\eta}_{ph}|^2_{BJS}\Big)\, ds \,\leq\, 0\,,$$

so it follows that $p_{ph}(t) = 0$, $\boldsymbol{\eta}_{ph}(t) = 0$, $\mathbf{u}_{fh}(t) = 0$, and $\mathbf{u}_{ph}(t) = 0$. Next, combining the inf-sup condition (3.3) in Lemma 3.1 with (3.4a) and (3.4d), we deduce that $p_{fh}(t) = 0$ and $\lambda_h(t) = 0$, concluding the uniqueness proof.

**Existence and uniqueness of a solution to (LGP2).** Next, we establish solvability of the transport equation (3.5), which leads to the system (3.8). Since the matrix $M_c$ is invertible, we can rewrite (3.8) as

$$\frac{d}{dt}\bar{c}(t) = M_c^{-1}\mathcal{F}_c(t) - M_c^{-1}\tilde{D}\,\bar{c}(t) := g(t, \bar{c}(t))\,.$$

Note that $g(t, \bar{c}(t))$ is continuous in $t$, due to the smoothness assumption on $q^-$, $q^+c_w$, and $\boldsymbol{\theta}$, and it is linear in $\bar{c}$. Therefore, it follows from the ODE theory, see e.g. Picard–Lindelöf theorem, [29, Theorem 1.3.1], that there exists a unique maximal solution $\bar{c}(t)$ in $(0, T]$ satisfying $c_h(0) = c_{h,0}$. □

### 3.3 Stability of (LGP1) and (LGP2)

In the stability analysis we will use the following inequality.

**Lemma 3.3** *Suppose that for all $t \in (0, T]$,*

$$\chi^2(t) + R(t) \leq A(t) + 2\int_0^t B(s)\chi(s)\, ds\,, \tag{3.20}$$

*where $\chi$, $R$, $A$ and $B$ are non-negative functions. Then for all $t \in (0, T]$,*

$$\sqrt{\chi^2(t) + R(t)} \leq \sup_{0\leq s\leq T}\sqrt{A(s)} + \int_0^t B(s)\, ds\,. \tag{3.21}$$

*Proof.* Let $\alpha(t) = \tilde{A} + 2\displaystyle\int_0^t B(s)\chi(s)ds$, where $\tilde{A} = \sup_{0\leq s\leq T} A(s)$. Then

$$\alpha'(t) = 2B(t)\chi(t) \leq 2B(t)\sqrt{\alpha(t)}\,,$$

implying $\big(\sqrt{\alpha(t)}\big)' \leq B(t)$. Integrating 0 to $t$ gives

$$\sqrt{\alpha(t)} \leq \sqrt{\alpha(0)} + \int_0^t B(s)ds = \sqrt{\tilde{A}} + \int_0^t B(s)\, ds\,.$$

Since (3.20) gives $\chi^2(t) + R(t) \leq \alpha(t)$, the above inequality implies (3.21). □

The stability bound mentioned above is now stated.

**Theorem 3.4** *Assume that* $\mathbf{f}_f \in L^\infty(0,T;\mathbf{V}_f')$, $\mathbf{f}_p \in H^1(0,T;\mathbf{X}_p')$, $q_f \in L^\infty(0,T;W_f')$, $q_p \in L^\infty(0,T;W_p')$, $\gamma \in L^\infty(0,T;L^\infty(\Omega))$, *and* $p_{p,0} \in H_p$. *Then, for the solution of* (**LGP1**), *there exists a positive constant* $C$ *independent of* $h$ *and* $s_0$ *such that*

$$\begin{aligned}
&\sqrt{s_0}\,\|p_{ph}\|_{L^\infty(0,T;W_p)} + \|\boldsymbol{\eta}_{ph}\|_{L^\infty(0,T;\mathbf{X}_p)} + \|\mathbf{u}_{fh}\|_{L^2(0,T;\mathbf{V}_f)} + \|p_{fh}\|_{L^2(0,T;W_f)} \\
&\quad + \|\mathbf{u}_{ph}\|_{L^2(0,T;\mathbf{L}^2(\Omega_p))} + \|p_{ph}\|_{L^2(0,T;W_p)} + \|\lambda_h\|_{L^2(0,T;\Lambda)} + |\mathbf{u}_{fh} - \partial_t\boldsymbol{\eta}_{ph}|_{L^2(0,T;BJS)} \\
&\leq C\Big(\|\mathbf{f}_f\|_{L^2(0,T;\mathbf{V}_f')} + \|\mathbf{f}_p\|_{L^2(0,T;\mathbf{X}_p')} + \|\partial_t\mathbf{f}_p\|_{L^1(0,T;\mathbf{X}_p')} + \|q_p\|_{L^2(0,T;W_p')} \\
&\quad + \|q_f\|_{L^2(0,T;W_f')} + \|p_{p,0}\|_{H^1(\Omega_p)} + \big\|\mu(c_0)^{-1}K\nabla p_{p,0}\big\|_{\mathbf{H}^1(\Omega_p)}\Big). 
\end{aligned} \tag{3.22}$$

*Moreover, if* $\mathbf{f}_f \in H^1(0,T;\mathbf{V}_f')$, $\mathbf{f}_p \in H^2(0,T;\mathbf{X}_p')$, $q_f \in H^1(0,T;W_f')$, $q_p \in H^1(0,T;W_p')$, *and* $\gamma \in W^{1,\infty}(0,T;L^\infty(\Omega))$, *it holds that*

$$\begin{aligned}
&\sqrt{s_0}\|\partial_t p_{ph}\|_{L^\infty(0,T;W_p)} + \|\partial_t\boldsymbol{\eta}_{ph}\|_{L^\infty(0,T;\mathbf{X}_p)} + \|\partial_t\mathbf{u}_{fh}\|_{L^2(0,T;\mathbf{V}_f)} + \|\partial_t p_{fh}\|_{L^2(0,T;W_f)} \\
&\quad + \|\partial_t\mathbf{u}_{ph}\|_{L^2(0,T;\mathbf{L}^2(\Omega_p))} + \|\nabla\cdot\mathbf{u}_{ph}\|_{L^\infty(0,T;L^2(\Omega_p))} + \|\partial_t p_{ph}\|_{L^2(0,T;W_p)} \\
&\quad + \|\partial_t\lambda_h\|_{L^2(0,T;\Lambda)} + |\partial_t\mathbf{u}_{fh} - \partial_{tt}\boldsymbol{\eta}_{ph}|_{L^2(0,T;BJS)} \\
&\leq C\Big(\|\mathbf{f}_f\|_{H^1(0,T;\mathbf{V}_f')} + \|\mathbf{f}_p\|_{H^1(0,T;\mathbf{X}_p')} + \|\partial_{tt}\mathbf{f}_p\|_{L^1(0,T;\mathbf{X}_p')} + \|q_f\|_{H^1(0,T;W_f')} + \|q_p\|_{H^1(0,T;W_p')} \\
&\quad + \frac{1}{\sqrt{s_0}}\big(\|p_{p,0}\|_{H^1(\Omega_p)} + \big\|\mu(c_0)^{-1}K\nabla p_{p,0}\big\|_{\mathbf{H}^1(\Omega_p)} + \|\mathbf{f}_f(0)\|_{\mathbf{V}_f'} + \|q_f(0)\|_{W_f'} + \|q_p(0)\|_{W_p'}\big)\Big).
\end{aligned} \tag{3.23}$$

*Assume that* $q^- \in L^\infty(0,T;L^\infty(\Omega))$, $q^+c_w \in L^\infty(0,T;X_c')$, $\boldsymbol{\theta} \in L^\infty(0,T;\mathbf{L}^\infty(\Omega))$, *and* $c_0 \in H^1(\Omega)$. *Then, for the solution of* (**LGP2**), *it holds that*

$$\|c_h\|_{L^\infty(0,T;L^2(\Omega))} + \|c_h\|_{L^2(0,T;X_c)} \leq C\,\sqrt{\exp(LT)}\,\Big(\|c_0\|_{H^1(\Omega)} + \|q^+c_w\|_{L^2(0,T;X_c')}\Big), \tag{3.24}$$

*where* $L = \|\phi^{-1/2}\mathbf{D}(\boldsymbol{\theta})^{-1/2}\boldsymbol{\theta}\|^2_{L^\infty(0,T;\mathbf{L}^\infty(\Omega))}$. *Furthermore, if* $q^+c_w \in L^\infty(0,T;L^2(\Omega))$ *and* $\boldsymbol{\theta} \in W^{1,\infty}(0,T;\mathbf{L}^\infty(\Omega))$, *it holds that*

$$\|\partial_t c_h\|_{L^2(0,T;L^2(\Omega))} + \|c_h\|_{L^\infty(0,T;X_c)} \leq C\,\sqrt{\exp(LT)}\,\Big(\|c_0\|_{H^1(\Omega)} + \|q^+c_w\|_{L^2(0,T;L^2(\Omega))}\Big). \tag{3.25}$$

*Proof.* Taking $(\mathbf{v}_{fh}, w_{fh}, \boldsymbol{\xi}_{ph}, \mathbf{v}_{ph}, w_{ph}, \nu_h) = (\mathbf{u}_{fh}, p_{fh}, \partial_t\boldsymbol{\eta}_{ph}, \mathbf{u}_{ph}, p_{ph}, \lambda_h)$ in (3.4a)–(3.4f), summing the equations, and integrating from 0 to $t \in (0,T]$ gives

$$\begin{aligned}
&\frac{s_0}{2}\|p_{ph}(t)\|^2_{W_p} + \frac{1}{2}a_p^e(\boldsymbol{\eta}_{ph}(t), \boldsymbol{\eta}_{ph}(t)) \\
&\quad + \int_0^t \Big(a_f(\mathbf{u}_{fh}, \mathbf{u}_{fh}; \gamma) + a_p^d(\mathbf{u}_{ph}, \mathbf{u}_{ph}; \gamma) + a_{BJS}(\mathbf{u}_{fh}, \partial_t\boldsymbol{\eta}_{ph}; \mathbf{u}_{fh}, \partial_t\boldsymbol{\eta}_{ph})\Big)ds \\
&= \frac{s_0}{2}\,\|p_{ph}(0)\|^2_{W_p} + \frac{1}{2}\,a_p^e(\boldsymbol{\eta}_{ph}(0), \boldsymbol{\eta}_{ph}(0)) \\
&\quad + \int_0^t \Big((\mathbf{f}_f, \mathbf{u}_{fh})_{\Omega_f} + (\mathbf{f}_p, \partial_t\boldsymbol{\eta}_{ph})_{\Omega_p} + (q_p, p_{ph})_{\Omega_p} + (q_f, p_{fh})_{\Omega_f}\Big)ds.
\end{aligned} \tag{3.26}$$

For the term $\displaystyle\int_0^t (\mathbf{f}_p, \partial_t\boldsymbol{\eta}_{ph})_{\Omega_p}\,ds$ we employ the identity

$$\int_0^t (\mathbf{f}_p, \partial_t\boldsymbol{\eta}_{ph})_{\Omega_p}\,ds = (\mathbf{f}_p, \boldsymbol{\eta}_{ph})_{\Omega_p}\Big|_0^t - \int_0^t (\partial_t\mathbf{f}_p, \boldsymbol{\eta}_{ph})_{\Omega_p}\,ds. \tag{3.27}$$

We combine (3.26) and (3.27) and use the coercivity and continuity of $a_f$, $a_p^d$, $a_p^e$ and $a_{BJS}$ from Lemma 2.1 and the Cauchy–Schwarz and Young's inequalities to obtain, for $\epsilon > 0$,

$$
\begin{aligned}
& s_0\|p_{ph}(t)\|_{W_p}^2 + \|\boldsymbol{\eta}_{ph}(t)\|_{\mathbf{X}_p}^2 + \int_0^t \Big( \|\mathbf{u}_{fh}\|_{\mathbf{V}_f}^2 + \|\mathbf{u}_{ph}\|_{\mathbf{L}^2(\Omega_p)}^2 + |\mathbf{u}_{fh} - \partial_t \boldsymbol{\eta}_{ph}|_{BJS}^2 \Big) ds \\
& \leq C \left( \|p_{ph}(0)\|_{W_p}^2 + \|\boldsymbol{\eta}_{ph}(0)\|_{\mathbf{X}_p}^2 + \|\mathbf{f}_p(0)\|_{\mathbf{X}_p'}^2 + \int_0^t \|\partial_t \mathbf{f}_p\|_{\mathbf{X}_p'} \|\boldsymbol{\eta}_{ph}\|_{\mathbf{X}_p} \, ds \right) \\
& \quad + \frac{C}{\epsilon}\|\mathbf{f}_p(t)\|_{\mathbf{X}_p'}^2 + \frac{C}{\epsilon}\int_0^t \Big( \|\mathbf{f}_f\|_{\mathbf{V}_f'}^2 + \|q_p\|_{W_p'}^2 + \|q_f\|_{W_f'}^2 \Big) ds \\
& \quad + \epsilon \|\boldsymbol{\eta}_{ph}(t)\|_{\mathbf{X}_p}^2 + \epsilon \int_0^t \Big( \|\mathbf{u}_{fh}\|_{\mathbf{V}_f}^2 + \|p_{ph}\|_{W_p}^2 + \|p_{fh}\|_{W_f}^2 \Big) ds \,. 
\end{aligned} \tag{3.28}
$$

In turn, by using the inf-sup condition (3.3), combined with (3.4a) and (3.4d), we deduce

$$
\begin{aligned}
& \beta_2 \, \|(p_{fh}, p_{ph}, \lambda_h)\|_{W_f \times W_p \times \Lambda} \leq \sup_{0 \neq (\mathbf{v}_{fh}, \mathbf{v}_{ph}) \in \mathbf{V}_f \times \mathbf{V}_p} \frac{b_f(\mathbf{v}_{fh}, p_{fh}) + b_p(\mathbf{v}_{ph}, p_{ph}) + b_\Gamma(\mathbf{v}_{fh}, \mathbf{v}_{ph}, 0; \lambda_h)}{\|(\mathbf{v}_{fh}, \mathbf{v}_{ph})\|_{\mathbf{V}_f \times \mathbf{V}_p}} \\
& = \sup_{0 \neq (\mathbf{v}_{fh}, \mathbf{v}_{ph}) \in \mathbf{V}_f \times \mathbf{V}_p} \frac{(\mathbf{f}_f, \mathbf{v}_{fh})_{\Omega_f} - a_f(\mathbf{u}_{fh}, \mathbf{v}_{fh}; \gamma) - a_p^d(\mathbf{u}_{ph}, \mathbf{v}_{ph}; \gamma) - a_{BJS}(\mathbf{u}_{fh}, \partial_t \boldsymbol{\eta}_{ph}; \mathbf{v}_{fh}, 0)}{\|(\mathbf{v}_{fh}, \mathbf{v}_{ph})\|_{\mathbf{V}_f \times \mathbf{V}_p}} \\
& \leq C \left( \|\mathbf{f}_f\|_{\mathbf{V}_f'} + \|\mathbf{u}_{fh}\|_{\mathbf{V}_f} + \|\mathbf{u}_{ph}\|_{\mathbf{L}^2(\Omega_p)} + |\mathbf{u}_{fh} - \partial_t \boldsymbol{\eta}_{ph}|_{BJS} \right) ,
\end{aligned} \tag{3.29}
$$

where we used the continuity bounds from Lemma 2.1 and the trace inequality (3.16). This yields

$$
\begin{aligned}
& \int_0^t \Big( \|p_{fh}\|_{W_f}^2 + \|p_{ph}\|_{W_p}^2 + \|\lambda_h\|_\Lambda^2 \Big) ds \\
& \quad \leq C \int_0^t \Big( \|\mathbf{f}_f\|_{\mathbf{V}_f'}^2 + \|\mathbf{u}_{fh}\|_{\mathbf{V}_f}^2 + \|\mathbf{u}_{ph}\|_{\mathbf{L}^2(\Omega_p)}^2 + |\mathbf{u}_{fh} - \partial_t \boldsymbol{\eta}_{ph}|_{BJS}^2 \Big) ds \,.
\end{aligned} \tag{3.30}
$$

Combining (3.30) with (3.28) and taking $\epsilon$ small enough, we obtain

$$
\begin{aligned}
& s_0 \, \|p_{ph}(t)\|_{W_p}^2 + \|\boldsymbol{\eta}_{ph}(t)\|_{\mathbf{X}_p}^2 \\
& \quad + \int_0^t \Big( \|\mathbf{u}_{fh}\|_{\mathbf{V}_f}^2 + \|\mathbf{u}_{ph}\|_{\mathbf{L}^2(\Omega_p)}^2 + |\mathbf{u}_{fh} - \partial_t \boldsymbol{\eta}_{ph}|_{BJS}^2 + \|p_{fh}\|_{W_f}^2 + \|p_{ph}\|_{W_p}^2 + \|\lambda_h\|_\Lambda^2 \Big) ds \\
& \leq C \Bigg( \int_0^t \Big( \|\mathbf{f}_f\|_{\mathbf{V}_f'}^2 + \|\partial_t \mathbf{f}_p\|_{\mathbf{X}_p'}^2 + \|q_p\|_{W_p'}^2 + \|q_f\|_{W_f'}^2 \Big) ds + \|\mathbf{f}_p(t)\|_{\mathbf{X}_p'} \\
& \quad + \|p_{ph}(0)\|_{W_p}^2 + \|\boldsymbol{\eta}_{ph}(0)\|_{\mathbf{X}_p}^2 + \|\mathbf{f}_p(0)\|_{\mathbf{X}_p'}^2 + \int_0^t \|\partial_t \mathbf{f}_p\|_{\mathbf{X}_p'} \|\boldsymbol{\eta}_{ph}\|_{\mathbf{X}_p} \, ds \Bigg) .
\end{aligned} \tag{3.31}
$$

To bound the initial values, recall that $(\mathbf{u}_{fh}(0), p_{fh}(0), \boldsymbol{\eta}_{ph}(0), \mathbf{u}_{ph}(0), p_{ph}(0), \lambda_h(0)) = (\mathbf{u}_{fh,0}, p_{fh,0}, \boldsymbol{\eta}_{ph,0}, \mathbf{u}_{ph,0}, p_{ph,0}, \lambda_{h,0})$, the initial data constructed in Theorem 3.2, cf. (3.18) and (3.19), where $\mathbf{u}_{sh,0} := I_{sh}\mathbf{u}_{s,0}$ is used as data. The continuity of the Scott–Zhang interpolant $I_{sh}$, bound (3.15), the definitions $\mathbf{u}_{p,0} = -\mu(c_0)^{-1} K \nabla p_{p,0}$ and $\lambda_0 = p_{p,0}|_{\Gamma_{fp}}$, and the stability of the Stokes problem (3.13) imply that

$$
\|\mathbf{u}_{sh,0}\|_{\mathbf{H}^1(\Omega_p)} \leq C \left( \|p_{p,0}\|_{H^1(\Omega_p)} + \|\mu(c_0)^{-1} K \nabla p_{p,0}\|_{\mathbf{H}^1(\Omega_p)} + \|\mathbf{f}_f(0)\|_{\mathbf{V}_f'} + \|q_f(0)\|_{W_f'} \right) . \tag{3.32}
$$

Using (3.32), the stability of the discrete Stokes–Darcy problem (3.18) [33, 42] and the discrete elasticity problem (3.19) imply

$$
\|\mathbf{u}_{fh}(0)\|_{\mathbf{V}_f}^2 + \|\mathbf{u}_{ph}(0)\|_{\mathbf{V}_p}^2 + \|\boldsymbol{\eta}_{ph}(0)\|_{\mathbf{X}_p}^2 + \|p_{fh}(0)\|_{W_f}^2 + \|p_{ph}(0)\|_{W_p}^2 + \|\lambda_h(0)\|_\Lambda^2
$$

$$\leq C\left(\|p_{p,0}\|^2_{H^1(\Omega_p)} + \|\mu(c_0)^{-1}K\nabla p_{p,0}\|^2_{\mathbf{H}^1(\Omega_p)} + \|\mathbf{f}_p(0)\|^2_{\mathbf{X}_p'} + \|\mathbf{f}_f(0)\|^2_{\mathbf{V}_f'} + \|q_f(0)\|^2_{W_f'}\right). \quad (3.33)$$

Combining (3.31) and (3.33) and using Lemma 3.3 for the term $\int_0^t \|\partial_t \mathbf{f}_p\|_{\mathbf{X}_p'}\|\boldsymbol{\eta}_{ph}\|_{\mathbf{X}_p}\,ds$ gives (3.22).

We note that (3.22) does not provide control on $\nabla\cdot\mathbf{u}_{ph}$, which is necessary for obtaining compactness for $\mathbf{u}_{ph}$ in $\mathbf{V}_p$ and passing to the limit. We next derive (3.23), assuming higher data regularity. This bound gives control on $\partial_t p_{ph}$ and $\partial_t\boldsymbol{\eta}_{ph}$, which allows us to bound $\nabla\cdot\mathbf{u}_{ph}$. In fact, differentiating in time (3.4a)–(3.4f) and taking $(\mathbf{v}_{fh}, w_{fh}, \boldsymbol{\xi}_{ph}, \mathbf{v}_{ph}, w_{ph}, \nu_h) = (\partial_t\mathbf{u}_{fh}, \partial_t p_{fh}, \partial_{tt}\boldsymbol{\eta}_{ph}, \partial_t\mathbf{u}_{ph}, \partial_t p_{ph}, \partial_t\lambda_h)$, we get

$$\begin{aligned}
&a_f(\partial_t\mathbf{u}_{fh}, \partial_t\mathbf{u}_{fh};\gamma) + a_p^d(\partial_t\mathbf{u}_{ph}, \partial_t\mathbf{u}_{ph};\gamma) + a_p^e(\partial_t\boldsymbol{\eta}_{ph}, \partial_{tt}\boldsymbol{\eta}_{ph})\\
&\quad + a_{BJS}(\partial_t\mathbf{u}_{fh}, \partial_{tt}\boldsymbol{\eta}_{ph}; \partial_t\mathbf{u}_{fh}, \partial_{tt}\boldsymbol{\eta}_{ph}) + s_0(\partial_{tt}p_{ph}, \partial_t p_{ph})_{\Omega_p}\\
&= (\partial_t q_f, \partial_t p_{fh})_{\Omega_f} + (\partial_t q_p, \partial_t p_{ph})_{\Omega_p} + (\partial_t\mathbf{f}_f, \partial_t\mathbf{u}_{fh})_{\Omega_f} + \partial_t(\partial_t\mathbf{f}_p, \partial_t\boldsymbol{\eta}_{ph})_{\Omega_p}\\
&\quad - (\partial_{tt}\mathbf{f}_p, \partial_t\boldsymbol{\eta}_{ph})_{\Omega_p} - (2\mu'(\gamma)\,\partial_t\gamma\,\boldsymbol{\epsilon}(\mathbf{u}_{fh}), \partial_t\boldsymbol{\epsilon}(\mathbf{u}_{fh}))_{\Omega_f} - (\mu'(\gamma)\,\partial_t\gamma\,K^{-1}\mathbf{u}_{ph}, \partial_t\mathbf{u}_{ph})_{\Omega_p}. 
\end{aligned} \quad (3.34)$$

Integrating (3.34) from 0 to $t\in(0,T]$ and using Lemma 2.1, (2.5), $\gamma\in W^{1,\infty}(0,T;L^\infty(\Omega))$, and the Cauchy–Schwarz and Young's inequalities, we deduce, for $\epsilon>0$,

$$\begin{aligned}
&s_0\|\partial_t p_{ph}(t)\|^2_{W_p} + \|\partial_t\boldsymbol{\eta}_{ph}(t)\|^2_{\mathbf{X}_p} + \int_0^t\left(\|\partial_t\mathbf{u}_{fh}\|^2_{\mathbf{V}_f} + \|\partial_t\mathbf{u}_{ph}\|^2_{\mathbf{L}^2(\Omega_p)} + |(\partial_t\mathbf{u}_{fh} - \partial_{tt}\boldsymbol{\eta}_{ph})|^2_{BJS}\right)ds\\
&\leq C\left(s_0\|\partial_t p_{ph}(0)\|^2_{W_p} + \|\partial_t\boldsymbol{\eta}_{ph}(0)\|^2_{\mathbf{X}_p} + \|\partial_t\mathbf{f}_p(0)\|^2_{\mathbf{X}_p'} + \int_0^t\|\partial_{tt}\mathbf{f}_p\|_{\mathbf{X}_p'}\|\partial_t\boldsymbol{\eta}_{ph}\|_{\mathbf{X}_p}ds\right)\\
&\quad + \epsilon\left(\int_0^t\left(\|\partial_t p_{fh}\|^2_{W_f} + \|\partial_t p_{ph}\|^2_{W_p} + \|\partial_t\mathbf{u}_{fh}\|^2_{\mathbf{V}_f} + \|\partial_t\mathbf{u}_{ph}\|^2_{\mathbf{L}^2(\Omega_p)}\right)ds + \|\partial_t\boldsymbol{\eta}_{ph}(t)\|^2_{\mathbf{X}_p}\right)\\
&\quad + \frac{C}{\epsilon}\left(\int_0^t\left(\|\partial_t q_f\|^2_{W_f'} + \|\partial_t q_p\|^2_{W_p'} + \|\partial_t\mathbf{f}_f\|^2_{\mathbf{V}_f'} + \|\mathbf{u}_{fh}\|^2_{\mathbf{V}_f} + \|\mathbf{u}_{ph}\|^2_{\mathbf{L}^2(\Omega_p)}\right)ds + \|\partial_t\mathbf{f}_p(t)\|^2_{\mathbf{X}_p'}\right).
\end{aligned} \quad (3.35)$$

In turn, applying the inf-sup condition (3.3) to bound $(\partial_t p_{fh}, \partial_t p_{ph}, \partial_t\lambda_h)$, and using the time-differentiated forms of (3.4a) and (3.4d), together with arguments as in (3.29)–(3.30), we derive

$$\begin{aligned}
\int_0^t\left(\|\partial_t p_{fh}\|^2_{W_f} + \|\partial_t p_{ph}\|^2_{W_p} + \|\partial_t\lambda_h\|^2_{\Lambda}\right)ds \,\leq\, C\int_0^t&\Big(\|\partial_t\mathbf{f}_f\|^2_{\mathbf{V}_f'} + \|\partial_t\mathbf{u}_{fh}\|^2_{\mathbf{V}_f}\\
+ \|\partial_t\mathbf{u}_{ph}\|^2_{\mathbf{L}^2(\Omega_p)} + |\partial_t\mathbf{u}_{fh} - \partial_{tt}\boldsymbol{\eta}_{ph}|^2_{BJS} &+ \|\mathbf{u}_{fh}\|^2_{\mathbf{V}_f} + \|\mathbf{u}_{ph}\|^2_{\mathbf{L}^2(\Omega_p)}\Big)\,ds,
\end{aligned} \quad (3.36)$$

where we also used (2.5) and the hypothesis $\gamma\in W^{1,\infty}(0,T;L^\infty(\Omega))$. Taking $\epsilon$ small enough in (3.35) and combining it with (3.36), we obtain

$$\begin{aligned}
&s_0\|\partial_t p_{ph}(t)\|^2_{W_p} + \|\partial_t\boldsymbol{\eta}_{ph}(t)\|^2_{\mathbf{X}_p} + \int_0^t\Big(\|\partial_t\mathbf{u}_{fh}\|^2_{\mathbf{V}_f} + \|\partial_t\mathbf{u}_{ph}\|^2_{\mathbf{L}^2(\Omega_p)} + |(\partial_t\mathbf{u}_{fh} - \partial_{tt}\boldsymbol{\eta}_{ph})|^2_{BJS}\\
&\quad + \|\partial_t p_{fh}\|^2_{W_f} + \|\partial_t p_{ph}\|^2_{W_p} + \|\partial_t\lambda_h\|^2_{\Lambda}\Big)ds\\
&\leq C\bigg(s_0\|\partial_t p_{ph}(0)\|^2_{W_p} + \|\partial_t\boldsymbol{\eta}_{ph}(0)\|^2_{\mathbf{X}_p} + \|\partial_t\mathbf{f}_p(0)\|^2_{\mathbf{X}_p'} + \int_0^t\|\partial_{tt}\mathbf{f}_p\|_{\mathbf{X}_p'}\|\partial_t\boldsymbol{\eta}_{ph}\|_{\mathbf{X}_p}ds\\
&\quad + \int_0^t\left(\|\partial_t q_f\|^2_{W_f'} + \|\partial_t q_p\|^2_{W_p'} + \|\partial_t\mathbf{f}_f\|^2_{\mathbf{V}_f'} + \|\mathbf{u}_{fh}\|^2_{\mathbf{V}_f} + \|\mathbf{u}_{ph}\|^2_{\mathbf{L}^2(\Omega_p)}\right)ds + \|\partial_t\mathbf{f}_p(t)\|^2_{\mathbf{X}_p'}\bigg).
\end{aligned} \quad (3.37)$$

We next focus on controlling the terms $\|\partial_t p_{ph}(0)\|_{W_p}$ and $\|\partial_t \boldsymbol{\eta}_{ph}(0)\|_{\mathbf{X}_p}$. Using (3.10) at $t=0$, we have $\partial_t \boldsymbol{\eta}_{ph}(0) = \mathbf{u}_{sh}(0) = \mathbf{u}_{sh,0}$. Then, (3.32) yields

$$\|\partial_t \boldsymbol{\eta}_{ph}(0)\|^2_{\mathbf{X}_p} \,\leq\, C\,\Big(\|p_{p,0}\|^2_{H^1(\Omega_p)} + \|\mu(c_0)^{-1}K\nabla p_{p,0}\|^2_{\mathbf{H}^1(\Omega_p)} + \|\mathbf{f}_f(0)\|^2_{\mathbf{V}_f'} + \|q_f(0)\|^2_{W_f'}\Big)\,. \tag{3.38}$$

To bound $\|\partial_t p_{ph}(0)\|_{W_p}$, we test (3.4e) at $t=0$ with $w_{ph} = \partial_t p_{ph}(0)$ and use (3.33), obtaining

$$\begin{aligned} &s_0\,\|\partial_t p_{ph}(0)\|^2_{W_p} \\ &\quad\leq\, C\,\Big(\|p_{p,0}\|^2_{H^1(\Omega_p)} + \|\mu(c_0)^{-1}K\nabla p_{p,0}\|^2_{\mathbf{H}^1(\Omega_p)} + \|\mathbf{f}_f(0)\|^2_{\mathbf{V}_f'} + \|q_f(0)\|^2_{W_f'} + \|q_p(0)\|^2_{W_p'}\Big)\,. \end{aligned} \tag{3.39}$$

Finally, to bound $\nabla\cdot\mathbf{u}_{ph}$, we test (3.4e) with $w_{ph} = \nabla\cdot\mathbf{u}_{ph}$, an admissible choice since, for a stable Darcy pair, $\nabla\cdot\mathbf{V}_{ph} = W_{ph}$. This leads to

$$\|\nabla\cdot\mathbf{u}_{ph}\|_{L^2(\Omega_p)} \,\leq\, C\,\Big(\|q_p\|_{W_p'} + s_0\|\partial_t p_{ph}\|_{W_p} + \|\partial_t\boldsymbol{\eta}_{ph}\|_{\mathbf{X}_p}\Big)\,. \tag{3.40}$$

Estimate (3.23) follows from combining (3.37)–(3.40) with (3.22) and using Lemma 3.3 for the term $\displaystyle\int_0^t \|\partial_{tt}\mathbf{f}_p\|_{\mathbf{X}_p'}\|\partial_t\boldsymbol{\eta}_{ph}\|_{\mathbf{X}_p}\,ds$.

For the transport equation, taking $\psi_h = c_h$ in equation (3.5), we have

$$\frac{1}{2}\frac{d}{dt}\|\phi^{1/2}c_h\|^2_{L^2(\Omega)} + \|\mathbf{D}(\boldsymbol{\theta})^{1/2}\nabla c_h\|^2_{L^2(\Omega)} - (c_h\boldsymbol{\theta},\nabla c_h) - (q^- c_h, c_h) \,=\, (q^+ c_w, c_h)\,. \tag{3.41}$$

Since $-(q^- c_h, c_h) \geq 0$, this term can be dropped. Moreover, using that $\boldsymbol{\theta}\in L^\infty(0,T;\mathbf{L}^\infty(\Omega))$, we deduce that

$$\begin{aligned} (c_h\boldsymbol{\theta},\nabla c_h) + (q^+ c_w, c_h) &\leq \frac{1}{2}\|c_h\mathbf{D}(\boldsymbol{\theta})^{-1/2}\boldsymbol{\theta}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\mathbf{D}(\boldsymbol{\theta})^{1/2}\nabla c_h\|^2_{L^2(\Omega)} + \epsilon\|c_h\|^2_{X_c} + \frac{C}{\epsilon}\|q^+c_w\|^2_{X_c'} \\ &\leq \frac{L}{2}\|\phi^{1/2}c_h\|^2_{L^2(\Omega)} + \frac{1}{2}\|\mathbf{D}(\boldsymbol{\theta})^{1/2}\nabla c_h\|^2_{L^2(\Omega)} + \epsilon\|c_h\|^2_{X_c} + \frac{C}{\epsilon}\|q^+c_w\|^2_{X_c'}\,, \end{aligned} \tag{3.42}$$

where $L = \|\phi^{-1/2}\mathbf{D}(\boldsymbol{\theta})^{-1/2}\boldsymbol{\theta}\|^2_{L^\infty(0,T;\mathbf{L}^\infty(\Omega))}$. Thus, using (2.13) and the Poincaré inequality in (3.41), and choosing $\epsilon$ sufficiently small in (3.42), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\phi^{1/2}c_h\|^2_{L^2(\Omega)} + C_1\|c_h\|^2_{X_c} \leq \frac{L}{2}\|\phi^{1/2}c_h\|^2_{L^2(\Omega)} + C_2\|q^+c_w\|^2_{X_c'}\,.$$

Integrating from 0 to $t\in(0,T]$ and using that $\phi\geq\phi_*>0$ and Gronwall's inequality, we arrive at

$$\|c_h(t)\|^2_{L^2(\Omega)} + \int_0^t \|c_h\|^2_{X_c}\,ds \,\leq\, C\,\exp(LT)\,\left(\|c_h(0)\|^2_{L^2(\Omega)} + \int_0^t \|q^+c_w\|^2_{X_c'}\,ds\right)\,, \tag{3.43}$$

which, combined with $c_h(0) = c_{h,0} = I_{ch}c_0$ and the continuity of $I_{ch}$ in $H^1(\Omega)$, yields (3.24).

It remains to prove the higher regularity bound (3.25). We take $\psi_h = \partial_t c_h$ in (3.5) to obtain

$$\begin{aligned} &(\phi\partial_t c_h, \partial_t c_h) + \frac{1}{2}\frac{d}{dt}(\mathbf{D}(\boldsymbol{\theta})\nabla c_h,\nabla c_h) - \frac{1}{2}(\partial_t\mathbf{D}(\boldsymbol{\theta})\nabla c_h,\nabla c_h) - \frac{d}{dt}(c_h\boldsymbol{\theta},\nabla c_h) \\ &\quad + (\partial_t c_h\boldsymbol{\theta},\nabla c_h) + (c_h\partial_t\boldsymbol{\theta},\nabla c_h) - (q^-c_h,\partial_t c_h) = (q^+c_w,\partial_t c_h)\,. \end{aligned}$$

Next, we integrate the above equation from 0 to $t\in(0,T]$ and use (2.13) and (2.14), the Cauchy–Schwarz and Young's inequalities, and $\boldsymbol{\theta}\in W^{1,\infty}(0,T;\mathbf{L}^\infty(\Omega))$, to deduce, for some $\epsilon>0$,

$$\int_0^t \|\partial_t c_h\|^2_{L^2(\Omega)}ds + \|\nabla c_h(t)\|^2_{\mathbf{L}^2(\Omega)}$$

$$\leq C\left(\|c_h(0)\|^2_{H^1(\Omega)} + \int_0^t \left(\|\nabla c_h\|^2_{\mathbf{L}^2(\Omega)} + \|c_h\|^2_{L^2(\Omega)}\right)ds\right) + \epsilon\left(\|\nabla c_h(t)\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t \|\partial_t c_h\|^2_{L^2(\Omega)}ds\right)$$
$$+ \frac{C}{\epsilon}\left(\|c_h(t)\|^2_{L^2(\Omega)} + \int_0^t \left(\|\nabla c_h\|^2_{\mathbf{L}^2(\Omega)} + \|c_h\|^2_{L^2(\Omega)} + \|q^+ c_w\|^2_{L^2(\Omega)}\right)ds\right).$$

Thus, taking $\epsilon$ small enough and using (3.43) gives (3.25). $\square$

### 3.4 Existence and uniqueness of a solution to (LP1) and (LP2)

Before stating the result, we recall that the initial conditions $p_p(0) = p_{p,0}$ and $c(0) = c_0$ are given. In the proof of Theorem 3.2, we constructed initial data for the remaining variables that satisfy the weak formulation at $t = 0$ (cf. (3.12)–(3.17)).

**Theorem 3.5** *Let $\mathbf{f}_f \in H^1(0,T;\mathbf{V}_f')$, $\mathbf{f}_p \in H^2(0,T;\mathbf{X}_p')$, $q_f \in H^1(0,T;W_f')$, $q_p \in H^1(0,T;W_p')$, $\gamma \in W^{1,\infty}(0,T;L^\infty(\Omega))$, and $p_{p,0} \in H_p$. Let the initial data for the rest of the variables be constructed in (3.12)–(3.17). Then there exists a unique solution to* (**LP1**)*, $\mathbf{u}_f \in H^1(0,T;\mathbf{V}_f)$, $p_f \in H^1(0,T;W_f)$, $\boldsymbol{\eta}_p \in W^{1,\infty}(0,T;\mathbf{X}_p)$, $\mathbf{u}_p \in H^1(0,T;\mathbf{L}^2(\Omega_p) \cap L^\infty(0,T;\mathbf{V}_p)$, $p_p \in W^{1,\infty}(0,T;W_p)$, $\lambda \in H^1(0,T;\Lambda)$, satisfying $p_p(0) = p_{p,0}$ and $\boldsymbol{\eta}_p(0) = \boldsymbol{\eta}_{p,0}$. Moreover, $\mathbf{u}_f(0) = \mathbf{u}_{f,0}$, $p_f(0) = p_{f,0}$, $\mathbf{u}_p(0) = \mathbf{u}_{p,0}$, and $\lambda(0) = \lambda_0$.*

*Let $q^- \in L^\infty(0,T;L^\infty(\Omega))$, $q^+ c_w \in L^\infty(0,T;X_c')$, $\boldsymbol{\theta} \in W^{1,\infty}(0,T;\mathbf{L}^\infty(\Omega))$, and $c_0 \in H^1(\Omega)$. Then, there exists a unique solution to* (**LP2**)*, $c \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;X_c)$, with $c(0) = c_0$.*

*Proof.* We pass to the limit in the sequences defined by (**LGP1**) and (**LGP2**). Theorem 3.4 implies that $\{\mathbf{u}_{fh}\}$ is bounded in $H^1(0,T;\mathbf{H}^1(\Omega_f))$, $\{p_{fh}\}$ is bounded in $H^1(0,T;L^2(\Omega_f))$, $\{\boldsymbol{\eta}_{ph}\}$ is bounded in $W^{1,\infty}(0,T;\mathbf{H}^1(\Omega_p))$, $\{\mathbf{u}_{ph}\}$ is bounded in $H^1(0,T;\mathbf{L}^2(\Omega_p)) \cap L^\infty(0,T;\mathbf{H}(\mathrm{div};\Omega_\mathrm{p}))$, $\{p_{ph}\}$ is bounded in $W^{1,\infty}(0,T;L^2(\Omega_p))$, $\{\lambda_h\}$ is bounded in $H^1(0,T;H^{1/2}(\Gamma_{fp}))$, and $\{c_h\}$ is bounded in $H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$. By the Eberlein–Šmulian theorem [28], there exist $\mathbf{u}_f \in H^1(0,T;\mathbf{H}^1(\Omega_f))$, $p_f \in H^1(0,T;L^2(\Omega_f))$, $\boldsymbol{\eta}_p \in W^{1,\infty}(0,T;\mathbf{H}^1(\Omega_p))$, $\mathbf{u}_p \in H^1(0,T;\mathbf{L}^2(\Omega_p)) \cap L^\infty(0,T;\mathbf{H}(\mathrm{div};\Omega_\mathrm{p}))$, $p_p \in W^{1,\infty}(0,T;L^2(\Omega_p))$, $\lambda \in H^1(0,T;H^{1/2}(\Gamma_{fp}))$, and $c \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$, such that, up to a subsequence,

$$\begin{aligned}
&\mathbf{u}_{fh} \rightharpoonup \mathbf{u}_f \text{ in } H^1(0,T;\mathbf{H}^1(\Omega_f)), && p_{fh} \rightharpoonup p_f \text{ in } H^1(0,T;L^2(\Omega_f)),\\
&\boldsymbol{\eta}_{ph} \rightharpoonup \boldsymbol{\eta}_p \text{ in } H^1(0,T;\mathbf{H}^1(\Omega_p)), && p_{ph} \rightharpoonup p_p \text{ in } H^1(0,T;L^2(\Omega_p)),\\
&\mathbf{u}_{ph} \rightharpoonup \mathbf{u}_p \text{ in } H^1(0,T;\mathbf{L}^2(\Omega_p)) \cap L^2(0,T;\mathbf{H}(\mathrm{div};\Omega_p)), && \lambda_h \rightharpoonup \lambda \text{ in } H^1(0,T;H^{1/2}(\Gamma_{fp})),\\
&c_h \rightharpoonup c \text{ in } L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)),
\end{aligned}$$

where $\rightharpoonup$ denotes weak convergence. Fixing a set of test functions $(\mathbf{v}_{fh}, w_{fh}, \boldsymbol{\xi}_{ph}, \mathbf{v}_{ph}, w_{ph}, \nu_h) \in C^0(0,T;\mathbf{V}_{fh} \times W_{fh} \times \mathbf{X}_{ph} \times \mathbf{V}_{ph} \times W_{ph} \times \Lambda_h)$ in (3.4) and $\psi_h \in C^0(0,T;X_{ch})$ in (3.5), integrating in time from 0 to $T$, and taking $h \to 0$, we conclude that $(\mathbf{u}_f, p_f, \boldsymbol{\eta}_p, \mathbf{u}_p, p_p, \lambda)$ and $c$ satisfy the time-integrated versions of (3.1) and (3.2), respectively, with this choice of test functions. Since the spaces $C^0(0,T;\mathbf{V}_{fh} \times W_{fh} \times \mathbf{X}_{ph} \times \mathbf{V}_{ph} \times W_{ph} \times \Lambda_h)$ and $C^0(0,T;X_{ch})$ are dense in $L^2(0,T;\mathbf{V}_f \times W_f \times \mathbf{X}_p \times \mathbf{V}_p \times W_p \times \Lambda)$ and $L^2(0,T;X_c)$, respectively, we deduce that (3.1) and (3.2) hold for a.e. $t \in (0,T]$. It remains to shows that the initial conditions are satisfied. For any $w_p \in W_p$ we write

$$(p_{p,0} - p_p(0), w_p)_{\Omega_p} = (p_{p,0} - p_{ph,0}, w_p)_{\Omega_p} + (p_{ph}(0) - p_p(0), w_p)_{\Omega_p},$$

where we used that $p_{ph}(0) = p_{ph,0}$. Note that $(p_{p,0} - p_{ph,0}, w_p)_{\Omega_p} \to 0$ as $h \to 0$, due to the approximation property of the Stokes–Darcy elliptic projection (3.18), and $(p_{ph}(0) - p_p(0), w_p)_{\Omega_p} \to$

0 from the weak convergence $p_{ph} \rightharpoonup p_p$ in $H^1(0,T;L^2(\Omega_p))$. Therefore $p_p(0) = p_{p,0}$. A similar argument shows that $\mathbf{u}_f(0) = \mathbf{u}_{f,0}$, $p_f(0) = p_{f,0}$, $\mathbf{u}_p(0) = \mathbf{u}_{p,0}$, and $\lambda(0) = \lambda_0$. In a similar way we conclude that $\boldsymbol{\eta}_p(0) = \boldsymbol{\eta}_{p,0}$ and $c(0) = c_0$, using the approximation properties of the elasticity elliptic projection (3.19) and the Scott–Zhang interpolant $I_{ch}$, respectively.

The uniqueness of the solution to (3.1) satisfying $p_p(0) = p_{p,0}$ and $\boldsymbol{\eta}_p(0) = \boldsymbol{\eta}_{p,0}$, and of the solution to (3.2) satisfying $c(0) = c_0$ follow from the uniqueness arguments in the proof of Theorem 3.2, using the continuous inf-sup condition (2.12). □

# 4 Analysis of the coupled non-linear problem (P)

The analysis of the original fully coupled non-linear problem (**P**) is done by considering an iteration of decoupled linearized problems (**LP1**) and (**LP2**) and utilizing a fixed point iteration argument to show that it converges. We first obtain a local in time solution, then extend it on small time intervals until the solution is obtained globally on $(0,T]$. A similar technique has been used in [12]. The iteration is as follows: given $c^0$ and $\mathbf{u}^0$, for $m \geq 1$, solve

(**LP1m**): Find $(\mathbf{u}_f^m, p_f^m, \boldsymbol{\eta}_p^m, \mathbf{u}_p^m, p_p^m, \lambda^m) : [0,T] \to \mathbf{V}_f \times W_f \times \mathbf{X}_p \times \mathbf{V}_p \times W_p \times \Lambda$ such that $p_p^m(0) = p_{p,0}$, $\boldsymbol{\eta}_p^m(0) = \boldsymbol{\eta}_{p,0}$, and for a.e. $t \in (0,T]$ and for all $\mathbf{v}_f \in \mathbf{V}_f$, $w_f \in W_f$, $\boldsymbol{\xi}_p \in \mathbf{X}_p$, $\mathbf{v}_p \in \mathbf{V}_p$, $w_p \in W_p$, and $\nu \in \Lambda$,

$$a_f(\mathbf{u}_f^m, \mathbf{v}_f; c^{m-1}) + b_f(\mathbf{v}_f, p_f^m) + a_{BJS}(\mathbf{u}_f^m, \partial_t \boldsymbol{\eta}_p^m; \mathbf{v}_f, 0) + b_\Gamma(\mathbf{v}_f, 0, 0; \lambda^m) = (\mathbf{f}_f, \mathbf{v}_f)_{\Omega_f}\,, \tag{4.1a}$$

$$-\, b_f(\mathbf{u}_f^m, w_f) = (q_f, w_f)_{\Omega_f}\,, \tag{4.1b}$$

$$a_p^e(\boldsymbol{\eta}_p^m, \boldsymbol{\xi}_p) + \alpha b_e(\boldsymbol{\xi}_p, p_p^m) + a_{BJS}(\mathbf{u}_f^m, \partial_t \boldsymbol{\eta}_p^m; 0, \boldsymbol{\xi}_p) + b_\Gamma(0, 0, \boldsymbol{\xi}_p; \lambda^m) = (\mathbf{f}_p, \boldsymbol{\xi}_p)_{\Omega_p}\,, \tag{4.1c}$$

$$a_p^d(\mathbf{u}_p^m, \mathbf{v}_p; c^{m-1}) + b_p(\mathbf{v}_p, p_p^m) + b_\Gamma(0, \mathbf{v}_p, 0; \lambda^m) = 0\,, \tag{4.1d}$$

$$s_0(\partial_t p_p^m, w_p)_{\Omega_p} - \alpha b_e(\partial_t \boldsymbol{\eta}_p^m, w_p) - b_p(\mathbf{u}_p^m, w_p) = (q_p, w_p)_{\Omega_p}\,, \tag{4.1e}$$

$$b_\Gamma(\mathbf{u}_f^m, \mathbf{u}_p^m, \partial_t \boldsymbol{\eta}_p^m; \nu) = 0\,, \tag{4.1f}$$

(**LP2m**): Find $c^m : [0,T] \to X_c$ such that $c^m(0) = c_0$, and for a.e. $t \in (0,T)$ and for all $\psi \in X_c$,

$$(\phi \partial_t c^m, \psi) + (\mathbf{D}(\mathbf{u}^{m-1}) \nabla c^m, \nabla \psi) - (c^m \mathbf{u}^{m-1}, \nabla \psi) - (q^- c^m, \psi) = (q^+ c_w, \psi)\,, \tag{4.2}$$

where $\mathbf{u}^m|_{\Omega_f} := \mathbf{u}_f^m$ and $\mathbf{u}^m|_{\Omega_p} := \mathbf{u}_p^m$. Recall that the initial conditions have been constructed in Theorem 3.2. The well-posedness of (**LP1m**) and (**LP2m**) can be established using Theorem 3.5. We make the following regularity assumption on the velocity $\mathbf{u}^m$ and the concentration $c^m$.

**Assumption 4.1** *For all $m \geq 0$, $\mathbf{u}_f^m \in W^{1,\infty}(0,T;\mathbf{W}^{1,\infty}(\Omega_f))$, $\mathbf{u}_p^m \in W^{1,\infty}(0,T;\mathbf{L}^\infty(\Omega_p))$, and $c^m \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega))$, with finite constants $\beta_{\mathbf{u}}$ and $\beta_c$, where*

$$\beta_{\mathbf{u}} := \max\left\{ \sup_m \|\mathbf{u}_f^m\|_{W^{1,\infty}(0,T;\mathbf{W}^{1,\infty}(\Omega_f))}, \sup_m \|\mathbf{u}_p^m\|_{W^{1,\infty}(0,T;\mathbf{L}^\infty(\Omega_p))} \right\}$$

$$\text{and} \quad \beta_c := \sup_m \|c^m\|_{W^{1,\infty}(0,T;W^{1,\infty}(\Omega))}\,.$$

The above assumption can be justified for sufficiently smooth data and domain boundary. In particular, the solution to (4.1) satisfies in distributional sense the partial differential equation system (2.1)–(2.2) with interface conditions (2.7). The Stokes solution can be eliminated from the system by solving (2.1) with boundary conditions on $\Gamma_{fp}$ given by the continuity of normal velocity condition, the first equation in (2.7a), and the BJS condition, the second equation in (2.7b). The Darcy velocity $\mathbf{u}_p$ can be eliminated using Darcy's law, the second equation in (2.2a). Finally, the displacement $\boldsymbol{\eta}_p$ can be eliminated by solving the elasticity problem given by the first equation in (2.2a) using the continuity of normal stress, the first equation in (2.7b), as boundary condition on $\Gamma_{fp}$. This results in $p_p^m(t)$ satisfying the parabolic equation

$$
\begin{aligned}
&s_0 \partial_t p_p^m + \mathcal{A}\partial_t p_p^m - \nabla \cdot \frac{K}{\mu(c^{m-1})}\nabla p_p^m = q_p \quad \text{in} \quad \Omega_p \times (0,T],\\
&p_p^m = 0 \text{ on } \Gamma_p^D \times (0,T], \quad \frac{K}{\mu(c^{m-1})}\nabla p_p^m \cdot \mathbf{n}_p = 0 \text{ on } \Gamma_p^N \times (0,T], \quad p_p^m = \mathcal{B}p_p^m \text{ on } \Gamma_{fp} \times (0,T],
\end{aligned}
$$

with $p_p^m(0) = p_{p,0}$ in $\Omega_p$, where $\mathcal{A}$ is a coercive linear operator obtained by solving for $\boldsymbol{\eta}_p^m$ in terms of $p_p^m$ using the elasticity equation and $\mathcal{B}p_p^m := -(\boldsymbol{\sigma}_f^m \mathbf{n}_f)\cdot \mathbf{n}_f$ on $\Gamma_{fp}$ is a linear operator. Increased regularity for $p_p^m$ in both time and space can be obtained utilizing parabolic regularity arguments, see Section 7.1 in [34]. This implies increased regularity for $\mathbf{u}_p^m$ from Darcy's law. Increased regularity for $\mathbf{u}_f^m$ and $\boldsymbol{\eta}_p^m$ can be obtained from their respective equations using elliptic regularity arguments, see Section 6.3 in [34]. Similarly, increased regularity for $c^m$ can be obtained using parabolic regularity. For further details we refer the reader to [12] where a similar assumption is made for a thermo-poroelastic system.

**Remark 4.1** *The well-posedness of* (**LP1m**) *and* (**LP2m**) *is ensured by Theorem 3.5 and Assumption 4.1. In particular,* $\mathbf{u}^0$ *and* $c^0$ *can be chosen to satisfy Assumption 4.1. Then by Theorem 3.5* (**LP1m**) *and* (**LP2m**) *for* $m = 1$ *are well posed. Let their solutions satisfy Assumption 4.1. The well-posedness of* (**LP1m**) *and* (**LP2m**) *for all* $m \geq 1$ *follows by induction. Therefore problems* (4.1) *and* (4.2) *define a unique iterative sequence. We further note that, according to Theorem 3.5, in addition to* $p_p^m(0) = p_{p,0}$, $\boldsymbol{\eta}_p^m(0) = \boldsymbol{\eta}_{p,0}$, *and* $c^m(0) = c_0$, *it holds that* $\mathbf{u}_f^m(0) = \mathbf{u}_{f,0}$, $p_f^m(0) = p_{f,0}$, $\mathbf{u}_p^m(0) = \mathbf{u}_{p,0}$, *and* $\lambda^m(0) = \lambda_0$ *for* $m \geq 1$, *where the initial data are constructed in Theorem 3.2.*

We arrive at the main result for the original non-linear problem (**P**).

**Theorem 4.1** *For each*

$$
\begin{aligned}
&\mathbf{f}_f \in H^1(0,T;\mathbf{V}_f'), \quad \mathbf{f}_p \in H^2(0,T;\mathbf{X}_p'), \quad q_f \in H^1(0,T;W_f'), \quad q_p \in H^1(0,T;W_p'),\\
&q^+ c_w \in L^2(0,T;X_c'), \quad q^- \in L^\infty(0,T;L^\infty(\Omega)), \quad p_{p,0} \in H_p, \quad \text{and} \quad c_0 \in H^1(\Omega),
\end{aligned}
$$

*and under Assumption 4.1, the iterative sequence* $(\mathbf{u}_f^m, p_f^m, \boldsymbol{\eta}_p^m, \mathbf{u}_p^m, p_p^m, \lambda^m, c^m)$ *from* (4.1)–(4.2) *is well defined and converges to a unique weak solution* $(\mathbf{u}_f, p_f, \boldsymbol{\eta}_p, \mathbf{u}_p, p_p, \lambda, c)$ *of* (**P**), *satisfying* $p_p(0) = p_{p,0}$, $\boldsymbol{\eta}_p(0) = \boldsymbol{\eta}_{p,0}$, *and* $c(0) = c_0$. *Moreover,* $\mathbf{u}_f(0) = \mathbf{u}_{f,0}$, $p_f(0) = p_{f,0}$, $\mathbf{u}_p(0) = \mathbf{u}_{p,0}$, $\lambda(0) = \lambda_0$, *and*

$$
\begin{aligned}
&(\mathbf{u}_f, \mathbf{u}_p, \boldsymbol{\eta}_p) \in L^\infty(0,T;\mathbf{V}_f) \times (L^\infty(0,T;\mathbf{L}^2(\Omega_p)) \cap L^2(0,T;\mathbf{V}_p)) \times H^1(0,T;\mathbf{X}_p);\\
&(p_f, p_p, \lambda) \in L^\infty(0,T;W_f) \times H^1(0,T;W_p) \times L^\infty(0,T;\Lambda);\\
&c \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;X_c).
\end{aligned}
$$

*Proof.* Under the theorem assumptions, Theorem 3.5 holds and the well-posedness of (**LP1m**) and (**LP2m**) follows from Remark 4.1. It remains to establish the convergence of the sequence. For $m \geq 2$, denote by $(\mathbf{e}_{\mathbf{u}_f}^m, e_{p_f}^m, \mathbf{e}_{\boldsymbol{\eta}_p}^m, \mathbf{e}_{\mathbf{u}_p}^m, e_{p_p}^m, e_\lambda^m, e_c^m)$ the difference between the solutions of (**LP1m**) and (**LP2m**) at the $(m)^{th}$ step and the $(m-1)^{th}$ step, namely $\mathbf{e}_{\mathbf{u}_f}^m := \mathbf{u}_f^m - \mathbf{u}_f^{m-1}$, with analogous notation for the remaining variables.

Subtracting (4.1)–(4.2) at the $m^{th}$ and $(m-1)^{th}$ steps, we obtain $(\mathbf{e}^m_{\mathbf{u}_f}, e^m_{p_f}, \mathbf{e}^m_{\boldsymbol{\eta}_p}, \mathbf{e}^m_{\mathbf{u}_p}, e^m_{p_p}, e^m_\lambda, e^m_c) : [0,T] \to \mathbf{V}_f \times W_f \times \mathbf{X}_p \times \mathbf{V}_p \times W_p \times \Lambda \times X_c$ such that for all $(\mathbf{v}_f, w_f, \boldsymbol{\xi}_p, \mathbf{v}_p, w_p, \nu, \psi) \in \mathbf{V}_f \times W_f \times \mathbf{X}_p \times \mathbf{V}_p \times W_p \times \Lambda \times X_c$ and for a.e. $t \in (0,T)$,

$$
\begin{aligned}
& a_f(\mathbf{e}^m_{\mathbf{u}_f}, \mathbf{v}_f; c^{m-1}) + \big(2(\mu(c^{m-1}) - \mu(c^{m-2}))\boldsymbol{\epsilon}(\mathbf{u}^{m-1}_f), \boldsymbol{\epsilon}(\mathbf{v}_f)\big)_{\Omega_f} + a_{BJS}(\mathbf{e}^m_{\mathbf{u}_f}, \partial_t \mathbf{e}^m_{\boldsymbol{\eta}_p}; \mathbf{v}_f, 0) \\
& \quad + b_f(\mathbf{v}_f, e^m_{p_f}) + b_\Gamma(\mathbf{v}_f, 0, 0; e^m_\lambda) = 0\,, && \text{(4.3a)} \\
& - b_f(\mathbf{e}^m_{\mathbf{u}_f}, w_f) = 0\,, && \text{(4.3b)} \\
& a^e_p(\mathbf{e}^m_{\boldsymbol{\eta}_p}, \boldsymbol{\xi}_p) + \alpha b_e(\boldsymbol{\xi}_p, e^m_{p_p}) + a_{BJS}(\mathbf{e}^m_{\mathbf{u}_f}, \partial_t \mathbf{e}^m_{\boldsymbol{\eta}_p}; 0, \boldsymbol{\xi}_p) + b_\Gamma(0, 0, \boldsymbol{\xi}_p; e^m_\lambda) = 0\,, && \text{(4.3c)} \\
& a^d_p(\mathbf{e}^m_{\mathbf{u}_p}, \mathbf{v}_p; c^{m-1}) + \big((\mu(c^{m-1}) - \mu(c^{m-2}))K^{-1}\mathbf{u}^{m-1}_p, \mathbf{v}_p\big)_{\Omega_p} + b_p(\mathbf{v}_p, e^m_{p_p}) \\
& \quad + b_\Gamma(0, \mathbf{v}_p, 0; e^m_\lambda) = 0\,, && \text{(4.3d)} \\
& s_0(\partial_t e^m_{p_p}, w_p)_{\Omega_p} - \alpha b_e(\partial_t \mathbf{e}^m_{\boldsymbol{\eta}_p}, w_p) - b_p(\mathbf{e}^m_{\mathbf{u}_p}, w_p) = 0\,, && \text{(4.3e)} \\
& b_\Gamma(\mathbf{e}^m_{\mathbf{u}_f}, \mathbf{e}^m_{\mathbf{u}_p}, \partial_t \mathbf{e}^m_{\boldsymbol{\eta}_p}; \nu) = 0\,, && \text{(4.3f)} \\
& (\phi \partial_t e^m_c, \psi) + (\mathbf{D}(\mathbf{u}^{m-1})\nabla e^m_c, \nabla\psi) + ((\mathbf{D}(\mathbf{u}^{m-1}) - \mathbf{D}(\mathbf{u}^{m-2}))\nabla c^{m-1}, \nabla\psi) \\
& \quad - (e^m_c \mathbf{u}^{m-1}, \nabla\psi) - (c^{m-1}\mathbf{e}^{m-1}_{\mathbf{u}}, \nabla\psi) - (q^- e^m_c, \psi) = 0\,. && \text{(4.3g)}
\end{aligned}
$$

According to Remark 4.1, all variables satisfy homogeneous initial conditions. The proof consists of seven steps: (1) bound $(\mathbf{e}^m_{\mathbf{u}_f}, e^m_{p_f}, \mathbf{e}^m_{\boldsymbol{\eta}_p}, \mathbf{e}^m_{\mathbf{u}_p}, e^m_{p_p}, e^m_\lambda)$ on a small time interval; (2) bound $e^m_c$ on a small time interval; (3) using these bounds, establish a local in time convergence using a fixed point argument; (4) obtain convergence for $(\mathbf{e}^m_{\mathbf{u}_f}, e^m_{p_f}, \mathbf{e}^m_{\boldsymbol{\eta}_p}, \mathbf{e}^m_{\mathbf{u}_p}, e^m_{p_p}, e^m_\lambda, e^m_c)$ in stronger norms; (5) establish existence of a solution on a small time interval; (6) extend the results globally on $(0,T]$; (7) prove that the solution is unique.

**Step 1: Bound $(\mathbf{e}^m_{\mathbf{u}_f}, e^m_{p_f}, \mathbf{e}^m_{\boldsymbol{\eta}_p}, \mathbf{e}^m_{\mathbf{u}_p}, e^m_{p_p}, e^m_\lambda)$ in terms of $e^{m-1}_c$.**

Taking $(\mathbf{v}_f, w_f, \boldsymbol{\xi}_p, \mathbf{v}_p, w_p, \nu) = (\mathbf{e}^m_{\mathbf{u}_f}, e^m_{p_f}, \partial_t\mathbf{e}^m_{\boldsymbol{\eta}_p}, \mathbf{e}^m_{\mathbf{u}_p}, e^m_{p_p}, e^m_\lambda)$ in (4.3a)–(4.3f), we deduce that

$$
\begin{aligned}
& \frac{s_0}{2}\frac{d}{dt}\|e^m_{p_p}\|^2_{L^2(\Omega_p)} + \frac{1}{2}\frac{d}{dt} a^e_p(\mathbf{e}^m_{\boldsymbol{\eta}_p}, \mathbf{e}^m_{\boldsymbol{\eta}_p}) \\
& + a_f(\mathbf{e}^m_{\mathbf{u}_f}, \mathbf{e}^m_{\mathbf{u}_f}; c^{m-1}) + a^d_p(\mathbf{e}^m_{\mathbf{u}_p}, \mathbf{e}^m_{\mathbf{u}_p}; c^{m-1}) + a_{BJS}(\mathbf{e}^m_{\mathbf{u}_f}, \partial_t\mathbf{e}^m_{\boldsymbol{\eta}_p}; \mathbf{e}^m_{\mathbf{u}_f}, \partial_t\mathbf{e}^m_{\boldsymbol{\eta}_p}) \\
& = \big(2(\mu(c^{m-2}) - \mu(c^{m-1}))\boldsymbol{\epsilon}(\mathbf{u}^{m-1}_f), \boldsymbol{\epsilon}(\mathbf{e}^m_{\mathbf{u}_f})\big)_{\Omega_f} + \big((\mu(c^{m-2}) - \mu(c^{m-1}))K^{-1}\mathbf{u}^{m-1}_p, \mathbf{e}^m_{\mathbf{u}_p}\big)_{\Omega_p}\,. \qquad (4.4)
\end{aligned}
$$

Using Assumption 4.1 to control $\|\boldsymbol{\epsilon}(\mathbf{u}^{m-1}_f)\|_{L^\infty(0,T;\mathbf{L}^\infty(\Omega_f))}$ and the Lipschitz continuity of the viscosity function $\mu(\cdot)$ (cf. (2.6)), we have

$$
\big(2(\mu(c^{m-2}) - \mu(c^{m-1}))\boldsymbol{\epsilon}(\mathbf{u}^{m-1}_f), \boldsymbol{\epsilon}(\mathbf{e}^m_{\mathbf{u}_f})\big)_{\Omega_f} \,\le\, C\,\beta_{\mathbf{u}}\,\|c^{m-1} - c^{m-2}\|_{L^2(\Omega_f)}\|\boldsymbol{\epsilon}(\mathbf{e}^m_{\mathbf{u}_f})\|_{\mathbf{L}^2(\Omega_f)}\,. \qquad (4.5)
$$

Similarly, for the second term on the right hand side of (4.4) we obtain

$$
\big((\mu(c^{m-2}) - \mu(c^{m-1}))K^{-1}\mathbf{u}^{m-1}_p, \mathbf{e}^m_{\mathbf{u}_p}\big)_{\Omega_p} \,\le\, C\,\beta_{\mathbf{u}}\,\|c^{m-1} - c^{m-2}\|_{L^2(\Omega_p)}\|\mathbf{e}^m_{\mathbf{u}_p}\|_{\mathbf{L}^2(\Omega_p)}\,. \qquad (4.6)
$$

Combining (4.4)–(4.6), using the coercivity of $a_f$, $a^d_p$, $a^e_p$, and $a_{BJS}$ from Lemma 2.1, and integrating over time from 0 to $t$, as well as utilizing the Cauchy–Schwarz and Young's inequalities for some positive $\epsilon > 0$, we get

$$
\|e^m_{p_p}(t)\|^2_{L^2(\Omega_p)} + \|\mathbf{e}^m_{\boldsymbol{\eta}_p}(t)\|^2_{\mathbf{H}^1(\Omega_p)} + \int_0^t \Big(\|\mathbf{e}^m_{\mathbf{u}_f}\|^2_{\mathbf{H}^1(\Omega_f)} + \|\mathbf{e}^m_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} + |\mathbf{e}^m_{\mathbf{u}_f} - \partial_t\mathbf{e}^m_{\boldsymbol{\eta}_p}|^2_{BJS}\Big)\,ds
$$

$$\leq C \int_0^t \left( \|e_c^{m-1}\|^2_{L^2(\Omega_f)} + \|e_c^{m-1}\|^2_{L^2(\Omega_p)} \right) ds + \epsilon \int_0^t \left( \|\mathbf{e}^m_{\mathbf{u}_f}\|^2_{\mathbf{H}^1(\Omega_f)} + \|\mathbf{e}^m_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} \right) ds\,. \tag{4.7}$$

Next, using the inf-sup condition (2.12), (4.3a), (4.3d), the continuity of $a_f$, $a_p^d$, and $a_{BJS}$ from Lemma 2.1, and Assumption 4.1, we deduce that

$$\begin{aligned}
&\|(e^m_{p_f}, e^m_{p_p}, e^m_\lambda)\|_{W_f \times W_p \times \Lambda} \\
&\leq C\beta_{\mathbf{u}} \left( \|\mathbf{e}^m_{\mathbf{u}_f}\|_{\mathbf{H}^1(\Omega_f)} + \|\mathbf{e}^m_{\mathbf{u}_p}\|_{\mathbf{L}^2(\Omega_p)} + |\mathbf{e}^m_{\mathbf{u}_f} - \partial_t \mathbf{e}^m_{\boldsymbol{\eta}_p}|_{BJS} + \|e_c^{m-1}\|_{L^2(\Omega_f)} + \|e_c^{m-1}\|_{L^2(\Omega_p)} \right),
\end{aligned} \tag{4.8}$$

which yields

$$\begin{aligned}
&\int_0^t \left( \|e^m_{p_f}\|^2_{L^2(\Omega_f)} + \|e^m_{p_p}\|^2_{L^2(\Omega_p)} + \|e^m_\lambda\|^2_\Lambda \right) ds \\
&\quad \leq \int_0^t \left( \|\mathbf{e}^m_{\mathbf{u}_f}\|^2_{\mathbf{H}^1(\Omega_f)} + \|\mathbf{e}^m_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} + |\mathbf{e}^m_{\mathbf{u}_f} - \partial_t \mathbf{e}^m_{\boldsymbol{\eta}_p}|^2_{BJS} + \|e_c^{m-1}\|^2_{L^2(\Omega)} \right) ds\,.
\end{aligned} \tag{4.9}$$

Thus, taking $\epsilon$ small enough in (4.7), and using (4.9), we find that

$$\begin{aligned}
&\|e^m_{p_p}(t)\|^2_{L^2(\Omega_p)} + \|\mathbf{e}^m_{\boldsymbol{\eta}_p}(t)\|^2_{\mathbf{H}^1(\Omega_p)} + \int_0^t \left( \|\mathbf{e}^m_{\mathbf{u}_f}\|^2_{\mathbf{H}^1(\Omega_f)} + \|\mathbf{e}^m_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} + |\mathbf{e}^m_{\mathbf{u}_f} - \partial_t \mathbf{e}^m_{\boldsymbol{\eta}_p}|^2_{BJS} \right) ds \\
&\quad + \int_0^t \left( \|e^m_{p_f}\|^2_{L^2(\Omega_f)} + \|e^m_{p_p}\|^2_{L^2(\Omega_p)} + \|e^m_\lambda\|^2_\Lambda \right) ds \,\leq\, C \int_0^t \|e_c^{m-1}\|^2_{L^2(\Omega)}\, ds\,.
\end{aligned} \tag{4.10}$$

**Step 2: Bound $e_c^m$ in terms of $\mathbf{e}_{\mathbf{u}}^{m-1}$.**

Taking $\psi = e_c^m$ in (4.3g), gives

$$\begin{aligned}
&(\phi \partial_t e_c^m, e_c^m) + (\mathbf{D}(\mathbf{u}^{m-1})\nabla e_c^m, \nabla e_c^m) - (q^- e_c^m, e_c^m) \\
&\quad = \left( (\mathbf{D}(\mathbf{u}^{m-2}) - \mathbf{D}(\mathbf{u}^{m-1}))\nabla c^{m-1}, \nabla e_c^m \right) + (\mathbf{u}^{m-1} e_c^m, \nabla e_c^m) + (\mathbf{e}_{\mathbf{u}}^{m-1} c^{m-1}, \nabla e_c^m)\,.
\end{aligned} \tag{4.11}$$

Note that the term $-(q^- e_c^m, e_c^m)$ on the left hand side of (4.11) is non-negative which can be dropped directly. Using (2.15), for the first term on the right hand side of (4.11) we have

$$\left( (\mathbf{D}(\mathbf{u}^{m-2}) - \mathbf{D}(\mathbf{u}^{m-1}))\nabla c^{m-1}, \nabla e_c^m \right) \leq C \|\nabla c^{m-1}\|_{\mathbf{L}^\infty(\Omega)} \|\mathbf{e}_{\mathbf{u}}^{m-1}\|_{\mathbf{L}^2(\Omega)} \|\nabla e_c^m\|_{\mathbf{L}^2(\Omega)}\,. \tag{4.12}$$

For the remaining terms of the right hand side of (4.11) we deduce the bound

$$\begin{aligned}
&(\mathbf{u}^{m-1} e_c^m, \nabla e_c^m) + (\mathbf{e}_{\mathbf{u}}^{m-1} c^{m-1}, \nabla e_c^m) \\
&\quad \leq \|\mathbf{u}^{m-1}\|_{\mathbf{L}^\infty(\Omega)} \|e_c^m\|_{L^2(\Omega)} \|\nabla e_c^m\|_{\mathbf{L}^2(\Omega)} + \|c^{m-1}\|_{L^\infty(\Omega)} \|\mathbf{e}_{\mathbf{u}}^{m-1}\|_{\mathbf{L}^2(\Omega)} \|\nabla e_c^m\|_{\mathbf{L}^2(\Omega)}\,.
\end{aligned} \tag{4.13}$$

Combining (4.11)–(4.13), and using Assumption 4.1 and Young's inequality, we obtain

$$\frac{d}{dt}\|e_c^m\|^2_{L^2(\Omega)} + \|\nabla e_c^m\|^2_{\mathbf{L}^2(\Omega)} \leq C \left( \beta_c \|\mathbf{e}_{\mathbf{u}}^{m-1}\|^2_{\mathbf{L}^2(\Omega)} + \beta_{\mathbf{u}} \|e_c^m\|^2_{L^2(\Omega)} \right). \tag{4.14}$$

Then, adding $\|e_c^m\|^2_{L^2(\Omega)}$ to both sides of (4.14), integrating in time from 0 to $t$, and applying Gronwall's inequality, yields

$$\|e_c^m(t)\|^2_{L^2(\Omega)} + \int_0^t \left( \|e_c^m\|^2_{L^2(\Omega)} + \|\nabla e_c^m\|^2_{\mathbf{L}^2(\Omega)} \right) ds \,\leq\, C\, \exp(T) \int_0^t \|\mathbf{e}_{\mathbf{u}}^{m-1}\|^2_{\mathbf{L}^2(\Omega)}\, ds\,. \tag{4.15}$$

**Step 3: Establish $(\mathbf{e}^m_{\mathbf{u}_f}, e^m_{p_f}, \mathbf{e}^m_{\boldsymbol{\eta}_p}, \mathbf{e}^m_{\mathbf{u}_p}, e^m_{p_p}, e^m_\lambda, e^m_c) \to \mathbf{0}$ as $m \to \infty$ in suitable norms on a small time interval.**

Considering (4.15) with $t = t_1$, integrating again from 0 to $t_1$, and using (4.10), we deduce

$$\begin{aligned} \int_0^{t_1} \|e^m_c\|^2_{L^2(\Omega)}\, ds \,&\leq\, t_1\, C\, \exp(T) \int_0^{t_1} \left( \|\mathbf{e}^{m-1}_{\mathbf{u}_f}\|^2_{\mathbf{L}^2(\Omega_f)} + \|\mathbf{e}^{m-1}_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} \right) ds \\ &\leq t_1\, C\, \exp(T) \int_0^{t_1} \|e^{m-2}_c\|^2_{L^2(\Omega)}\, ds\,, \end{aligned} \tag{4.16}$$

where the constant $C > 0$ is independent of the iteration step $m$ and the local time $t_1$. Let $L := C \exp(T)$. Choosing $t_1 = \frac{1}{2L}$, we conclude that $\|e^m_c\|_{L^2(0,t_1;L^2(\Omega))} \to 0$ as $m \to \infty$. The estimates (4.10), (4.15), and (4.16) imply that, as $m \to \infty$,

$$\begin{aligned} (\mathbf{e}^m_{\mathbf{u}_f}, \mathbf{e}^m_{\mathbf{u}_p}, \mathbf{e}^m_{\boldsymbol{\eta}_p}) \to \mathbf{0} \quad &\text{in} \quad L^2(0,t_1;\mathbf{V}_f) \times L^2(0,t_1;\mathbf{L}^2(\Omega_p)) \times L^\infty(0,t_1;\mathbf{X}_p)\,; \\ (e^m_{p_f}, e^m_{p_p}, e^m_\lambda) \to \mathbf{0} \quad &\text{in} \quad L^2(0,t_1;W_f) \times L^\infty(0,t_1;W_p) \times L^2(0,t_1;\Lambda)\,; \\ e^m_c \to 0 \quad &\text{in} \quad L^\infty(0,t_1;L^2(\Omega)) \cap L^2(0,t_1;X_c)\,. \end{aligned} \tag{4.17}$$

**Step 4: Obtain $(\mathbf{e}^m_{\mathbf{u}_f}, e^m_{p_f}, \mathbf{e}^m_{\boldsymbol{\eta}_p}, \mathbf{e}^m_{\mathbf{u}_p}, e^m_{p_p}, e^m_\lambda, e^m_c) \to \mathbf{0}$ in stronger norms and $\nabla\cdot\mathbf{e}^m_{\mathbf{u}_p} \to 0$.**

We next obtain convergence in stronger norms, as well as for $\nabla\cdot\mathbf{e}^m_{\mathbf{u}_p}$, which will allow us to take the limit $m \to \infty$ in (4.1) and (4.2). First, we note that Cauchy-Schwarz inequality implies

$$\left( \int_0^{t_1} \int_{\Omega_f} |\boldsymbol{\epsilon}(\mathbf{e}^m_{\mathbf{u}_f})|\, ds \right)^2 \leq T\,|\Omega_f| \int_0^{t_1} \|\boldsymbol{\epsilon}(\mathbf{e}^m_{\mathbf{u}_f})\|^2_{\mathbf{L}^2(\Omega_f)}\, ds \to 0 \text{ as } m \to \infty\,, \tag{4.18}$$

$$\left( \int_0^{t_1} \int_{\Omega_p} |\mathbf{e}^m_{\mathbf{u}_p}|\, ds \right)^2 \leq T\,|\Omega_p| \int_0^{t_1} \|\mathbf{e}^m_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)}\, ds \to 0 \text{ as } m \to \infty\,, \tag{4.19}$$

$$\left( \int_0^{t_1} \int_{\Omega} |\nabla e^m_c|\, ds \right)^2 \leq T\,|\Omega| \int_0^{t_1} \|\nabla e^m_c\|^2_{\mathbf{L}^2(\Omega)}\, ds \to 0 \text{ as } m \to \infty\,. \tag{4.20}$$

Next, differentiating (4.3a)–(4.3f) in time and choosing the test functions $(\mathbf{v}_f, w_f, \boldsymbol{\xi}_p, \mathbf{v}_p, w_p, \nu) = (\mathbf{e}^m_{\mathbf{u}_f}, \partial_t e^m_{p_f}, \partial_t \mathbf{e}^m_{\boldsymbol{\eta}_p}, \mathbf{e}^m_{\mathbf{u}_p}, \partial_t e^m_{p_p}, \partial_t e^m_\lambda)$, we obtain

$$\begin{aligned} &a_f(\partial_t \mathbf{e}^m_{\mathbf{u}_f}, \mathbf{e}^m_{\mathbf{u}_f}; c^{m-1}) + a^d_p(\partial_t \mathbf{e}^m_{\mathbf{u}_p}, \mathbf{e}^m_{\mathbf{u}_p}; c^{m-1}) + a^e_p(\partial_t \mathbf{e}^m_{\boldsymbol{\eta}_p}, \partial_t \mathbf{e}^m_{\boldsymbol{\eta}_p}) + a_{BJS}(\partial_t \mathbf{e}^m_{\mathbf{u}_f}, \partial_{tt} \mathbf{e}^m_{\boldsymbol{\eta}_p}; \mathbf{e}^m_{\mathbf{u}_f}, \partial_t \mathbf{e}^m_{\boldsymbol{\eta}_p}) \\ &\quad + s_0(\partial_t e^m_{p_p}, \partial_t e^m_{p_p})_{\Omega_p} = -\big(2\partial_t \mu(c^{m-1}) \boldsymbol{\epsilon}(\mathbf{e}^m_{\mathbf{u}_f}), \boldsymbol{\epsilon}(\mathbf{e}^m_{\mathbf{u}_f})\big)_{\Omega_f} - \big(\partial_t \mu(c^{m-1}) K^{-1} \mathbf{e}^m_{\mathbf{u}_p}, \mathbf{e}^m_{\mathbf{u}_p}\big)_{\Omega_p} \\ &\quad - \big(2\partial_t(\mu(c^{m-1}) - \mu(c^{m-2}))\boldsymbol{\epsilon}(\mathbf{u}^{m-1}_f), \boldsymbol{\epsilon}(\mathbf{e}^m_{\mathbf{u}_f})\big)_{\Omega_f} - \big(2(\mu(c^{m-1}) - \mu(c^{m-2}))\partial_t \boldsymbol{\epsilon}(\mathbf{u}^{m-1}_f), \mathbf{e}^m_{\mathbf{u}_f}\big)_{\Omega_f} \\ &\quad - \big(\partial_t(\mu(c^{m-1}) - \mu(c^{m-2}))K^{-1}\mathbf{u}^{m-1}_p, \mathbf{e}^m_{\mathbf{u}_p}\big)_{\Omega_p} - \big((\mu(c^{m-1}) - \mu(c^{m-2}))K^{-1}\partial_t \mathbf{u}^{m-1}_p, \mathbf{e}^m_{\mathbf{u}_p}\big)_{\Omega_p}\,. \end{aligned}$$

We integrate in time from 0 to $t_1$ and use the coercivity of $a_f$, $a^d_p$, $a^e_p$, and $a_{BJS}$ from Lemma 2.1 to get

$$\begin{aligned} &\|\mathbf{e}^m_{\mathbf{u}_f}(t_1)\|^2_{\mathbf{H}^1(\Omega_f)} + \|\mathbf{e}^m_{\mathbf{u}_p}(t_1)\|^2_{\mathbf{L}^2(\Omega_p)} + \big|(\mathbf{e}^m_{\mathbf{u}_f} - \partial_t \mathbf{e}^m_{\boldsymbol{\eta}_p})(t_1)\big|^2_{BJS} \\ &\quad + \int_0^{t_1} \left( s_0 \|\partial_t e_{p_p}\|^2_{L^2(\Omega_p)} + \|\partial_t \mathbf{e}^m_{\boldsymbol{\eta}_p}\|^2_{\mathbf{H}^1(\Omega_p)} \right) ds \\ &\leq C \int_0^{t_1} \Big( \big(2\partial_t \mu(c^{m-1}) \boldsymbol{\epsilon}(\mathbf{e}^m_{\mathbf{u}_f}), \boldsymbol{\epsilon}(\mathbf{e}^m_{\mathbf{u}_f})\big)_{\Omega_f} + \big(\partial_t \mu(c^{m-1}) K^{-1} \mathbf{e}^m_{\mathbf{u}_p}, \mathbf{e}^m_{\mathbf{u}_p}\big)_{\Omega_p} \end{aligned} \tag{4.21}$$

$$
+ 2\big(\partial_t(\mu(c^{m-2}) - \mu(c^{m-1}))\boldsymbol{\epsilon}(\mathbf{u}_f^{m-1}), \boldsymbol{\epsilon}(\mathbf{e}_{\mathbf{u}_f}^m)\big)_{\Omega_f} + 2\big((\mu(c^{m-2}) - \mu(c^{m-1}))\partial_t\boldsymbol{\epsilon}(\mathbf{u}_f^{m-1}), \mathbf{e}_{\mathbf{u}_f}^m\big)_{\Omega_f}
$$
$$
+ \big(\partial_t(\mu(c^{m-2}) - \mu(c^{m-1}))K^{-1}\mathbf{u}_p^{m-1}, \mathbf{e}_{\mathbf{u}_p}^m\big)_{\Omega_p} + \big((\mu(c^{m-2}) - \mu(c^{m-1}))K^{-1}\partial_t\mathbf{u}_p^{m-1}, \mathbf{e}_{\mathbf{u}_p}^m\big)_{\Omega_p}\Big)\, ds\,.
$$

For the first two terms on the right hand side of (4.21), using (2.5) and Assumption 4.1, we have

$$
\int_0^{t_1} \Big(\big(2\partial_t\mu(c^{m-1})\boldsymbol{\epsilon}(\mathbf{e}_{\mathbf{u}_f}^m), \boldsymbol{\epsilon}(\mathbf{e}_{\mathbf{u}_f}^m)\big)_{\Omega_f} + \big(\partial_t\mu(c^{m-1})K^{-1}\mathbf{e}_{\mathbf{u}_p}^m, \mathbf{e}_{\mathbf{u}_p}^m\big)_{\Omega_p}\Big)\, ds
$$
$$
\leq C\,\beta_c \int_0^{t_1} \Big(\|\mathbf{e}_{\mathbf{u}_f}^m\|_{\mathbf{H}^1(\Omega_f)}^2 + \|\mathbf{e}_{\mathbf{u}_p}^m\|_{\mathbf{L}^2(\Omega_p)}^2\Big)\, ds \to 0, \text{ as } m \to \infty\,. \tag{4.22}
$$

For the third and fifth terms on the right hand side of (4.21), using (2.5), Assumption 4.1, (4.18), and (4.19), we get

$$
\int_0^{t_1} \Big(\big(2\partial_t(\mu(c^{m-2}) - \mu(c^{m-1}))\boldsymbol{\epsilon}(\mathbf{u}_f^{m-1}), \boldsymbol{\epsilon}(\mathbf{e}_{\mathbf{u}_f}^m)\big)_{\Omega_f} + \big(\partial_t(\mu(c^{m-2}) - \mu(c^{m-1}))K^{-1}\mathbf{u}_p^{m-1}, \mathbf{e}_{\mathbf{u}_p}^m\big)_{\Omega_p}\Big)\, ds
$$
$$
\leq C\,\beta_c\,\beta_{\mathbf{u}} \left(\int_0^{t_1}\int_{\Omega_f} |\boldsymbol{\epsilon}(\mathbf{e}_{\mathbf{u}_f}^m)|\, ds + \int_0^{t_1}\int_{\Omega_p} |\mathbf{e}_{\mathbf{u}_p}^m|\, ds\right) \to 0, \text{ as } m \to \infty\,. \tag{4.23}
$$

For the fourth and sixth terms on the right hand side of (4.21), using (2.6) and Assumption 4.1, we find that

$$
\int_0^{t_1} \Big|\big(2(\mu(c^{m-2}) - \mu(c^{m-1}))\partial_t\boldsymbol{\epsilon}(\mathbf{u}_f^{m-1}), \mathbf{e}_{\mathbf{u}_f}^m\big)_{\Omega_f} + \big((\mu(c^{m-2}) - \mu(c^{m-1}))K^{-1}\partial_t\mathbf{u}_p^{m-1}, \mathbf{e}_{\mathbf{u}_p}^m\big)_{\Omega_p}\Big|\, ds
$$
$$
\leq C\,\beta_{\mathbf{u}} \int_0^{t_1} \Big(\|e_c^{m-1}\|_{L^2(\Omega)}^2 + \|\mathbf{e}_{\mathbf{u}_f}^m\|_{\mathbf{L}^2(\Omega_f)}^2 + \|\mathbf{e}_{\mathbf{u}_p}^m\|_{\mathbf{L}^2(\Omega_p)}^2\Big)\, ds \to 0, \text{ as } m \to \infty. \tag{4.24}
$$

Combining (4.22)–(4.24) with (4.21) we obtain

$$
\|\mathbf{e}_{\mathbf{u}_f}^m(t_1)\|_{\mathbf{H}^1(\Omega_f)}^2 + \|\mathbf{e}_{\mathbf{u}_p}^m(t_1)\|_{\mathbf{L}^2(\Omega_p)}^2 + |(\mathbf{e}_{\mathbf{u}_f}^m - \partial_t\mathbf{e}_{\boldsymbol{\eta}_p}^m)(t_1)|_{BJS}^2
$$
$$
+ \int_0^{t_1} \Big(s_0\|\partial_t e_{p_p}\|_{L^2(\Omega_p)}^2 + \|\partial_t\mathbf{e}_{\boldsymbol{\eta}_p}^m\|_{\mathbf{H}^1(\Omega_p)}^2\Big)\, ds \to 0, \text{ as } m \to \infty\,. \tag{4.25}
$$

Next, take $w_p = \nabla\cdot\mathbf{e}_{\mathbf{u}_p}^m$ in (4.3e) and integrate in time from 0 to $t_1$, which results in

$$
\int_0^{t_1} \|\nabla\cdot\mathbf{e}_{\mathbf{u}_p}^m\|_{L^2(\Omega_p)}^2 \leq C\int_0^{t_1} \Big(s_0\|\partial_t e_{p_p}\|_{L^2(\Omega_p)}^2 + \|\partial_t\mathbf{e}_{\boldsymbol{\eta}_p}^m\|_{\mathbf{H}^1(\Omega_p)}^2\Big)\, ds \to 0, \text{ as } m \to \infty\,, \tag{4.26}
$$

where we have used (4.25). In addition, (4.8) gives

$$
\|e_{p_f}^m(t_1)\|_{L^2(\Omega_f)}^2 + \|e_{p_p}^m(t_1)\|_{L^2(\Omega_p)}^2 + \|e_\lambda^m(t_1)\|_\Lambda^2 \leq C\Big(\|\mathbf{e}_{\mathbf{u}_f}^m(t_1)\|_{\mathbf{H}^1(\Omega_f)}^2 + \|\mathbf{e}_{\mathbf{u}_p}^m(t_1)\|_{\mathbf{L}^2(\Omega_p)}^2
$$
$$
+ \big|(\mathbf{e}_{\mathbf{u}_f}^m - \partial_t\mathbf{e}_{\boldsymbol{\eta}_p}^m)(t_1)\big|_{BJS}^2 + \|e_c^{m-1}(t_1)\|_{L^2(\Omega)}^2\Big) \to 0, \text{ as } m \to \infty\,. \tag{4.27}
$$

We proceed with the transport error equation (4.3g). Taking the test function $\psi = \partial_t e_c^m$ gives

$$
(\phi\partial_t e_c^m, \partial_t e_c^m) + (\mathbf{D}(\mathbf{u}^{m-1})\nabla e_c^m, \nabla\partial_t e_c^m) - (q^- e_c^m, \partial_t e_c^m)
$$
$$
= \big((\mathbf{D}(\mathbf{u}^{m-2}) - \mathbf{D}(\mathbf{u}^{m-1}))\nabla c^{m-1}, \nabla\partial_t e_c^m\big) + (\mathbf{u}^{m-1}e_c^m, \nabla\partial_t e_c^m) + (\mathbf{e}_{\mathbf{u}}^{m-1}c^{m-1}, \nabla\partial_t e_c^m)\,. \tag{4.28}
$$

For the second term on the left hand side of (4.28) we write

$$(\mathbf{D}(\mathbf{u}^{m-1})\nabla e_c^m, \nabla\partial_t e_c^m) = \frac{1}{2}\frac{d}{dt}(\mathbf{D}(\mathbf{u}^{m-1})\nabla e_c^m, \nabla e_c^m) - \frac{1}{2}(\partial_t \mathbf{D}(\mathbf{u}^{m-1})\nabla e_c^m, \nabla e_c^m)\,.$$

Using similar expressions for the terms on the right hand side in (4.28) we obtain

$$\begin{aligned}
&(\phi\partial_t e_c^m, \partial_t e_c^m) + \frac{1}{2}\frac{d}{dt}(\mathbf{D}(\mathbf{u}^{m-1})\nabla e_c^m, \nabla e_c^m) \\
&= \frac{1}{2}(\partial_t \mathbf{D}(\mathbf{u}^{m-1})\nabla e_c^m, \nabla e_c^m) + (q^- e_c^m, \partial_t e_c^m) + \frac{d}{dt}\big((\mathbf{D}(\mathbf{u}^{m-2}) - \mathbf{D}(\mathbf{u}^{m-1}))\nabla c^{m-1}, \nabla e_c^m\big) \\
&\quad - \big(\partial_t(\mathbf{D}(\mathbf{u}^{m-2}) - \mathbf{D}(\mathbf{u}^{m-1}))\nabla c^{m-1}, \nabla e_c^m\big) - \big((\mathbf{D}(\mathbf{u}^{m-2}) - \mathbf{D}(\mathbf{u}^{m-1}))\nabla\partial_t c^{m-1}, \nabla e_c^m\big) \\
&\quad + \frac{d}{dt}(\mathbf{u}^{m-1}e_c^m, \nabla e_c^m) - (\partial_t \mathbf{u}^{m-1}e_c^m, \nabla e_c^m) - (\mathbf{u}^{m-1}\partial_t e_c^m, \nabla e_c^m) \\
&\quad + \frac{d}{dt}(\mathbf{e}_{\mathbf{u}}^{m-1}c^{m-1}, \nabla e_c^m) - (\partial_t \mathbf{e}_{\mathbf{u}}^{m-1}c^{m-1}, \nabla e_c^m) - (\mathbf{e}_{\mathbf{u}}^{m-1}\partial_t c^{m-1}, \nabla e_c^m)\,. \qquad (4.29)
\end{aligned}$$

For the fourth term on the right hand side of (4.29), using (2.14) and Assumption 4.1, we have

$$\begin{aligned}
&\big|\big(\partial_t(\mathbf{D}(\mathbf{u}^{m-2}) - \mathbf{D}(\mathbf{u}^{m-1}))\nabla c^{m-1}, \nabla e_c^m\big)\big| \\
&\quad \le 5(\alpha_t + \alpha_l)(\|\partial_t \mathbf{u}^{m-2}\|_{\mathbf{L}^\infty(\Omega)} + \|\partial_t \mathbf{u}^{m-1}\|_{\mathbf{L}^\infty(\Omega)})\|\nabla c^{m-1}\|_{\mathbf{L}^\infty(\Omega)}\|\nabla e_c^m\|_{\mathbf{L}^1(\Omega)} \\
&\quad \le 10(\alpha_t + \alpha_l)\beta_{\mathbf{u}}\beta_c\|\nabla e_c^m\|_{\mathbf{L}^1(\Omega)}\,. \qquad (4.30)
\end{aligned}$$

Similarly, for the second to last term on the right hand side of (4.29) we find that

$$\begin{aligned}
\big|(\partial_t \mathbf{e}_{\mathbf{u}}^{m-1}c^{m-1}, \nabla e_c^m)\big| &\le \big(\|\partial_t \mathbf{u}^{m-2}\|_{\mathbf{L}^\infty(\Omega)} + \|\partial_t \mathbf{u}^{m-1}\|_{\mathbf{L}^\infty(\Omega)}\big)\,\|c^{m-1}\|_{L^\infty(\Omega)}\|\nabla e_c^m\|_{\mathbf{L}^1(\Omega)} \\
&\le 2\beta_{\mathbf{u}}\beta_c\|\nabla e_c^m\|_{\mathbf{L}^1(\Omega)}. \qquad (4.31)
\end{aligned}$$

Next, we integrate (4.29) in time from 0 to $t_1$, using the fact that $e_c^m(0) = 0$, together with (4.30) and (4.31). For the remaining terms on the right-hand side, we bound one of the factors in the $L^\infty(\Omega)$ norm and apply the Cauchy–Schwarz inequality to the other two. Using Assumption 4.1 and the regularity of $q^-$ to control the $L^\infty(\Omega)$ terms, as well as (2.14) for the first term and (2.15) for the third and fifth terms, we obtain

$$\begin{aligned}
&\int_0^{t_1}(\phi\partial_t e_c^m, \partial_t e_c^m)\,ds + \frac{1}{2}(\mathbf{D}(\mathbf{u}^{m-1})\nabla e_c^m, \nabla e_c^m)(t_1) \\
&\le C\Bigg(\beta_{\mathbf{u}}\int_0^{t_1}\|\nabla e_c^m\|^2_{\mathbf{L}^2(\Omega)}\,ds + \|q^-\|_{L^\infty(0,t_1;L^\infty(\Omega))}\int_0^{t_1}\|e_c^m\|_{L^2(\Omega)}\|\partial_t e_c^m\|_{L^2(\Omega)}\,ds \\
&\quad + \beta_c\|\mathbf{e}_{\mathbf{u}}^{m-1}(t_1)\|_{\mathbf{L}^2(\Omega)}\|\nabla e_c^m(t_1)\|_{\mathbf{L}^2(\Omega)} + \beta_{\mathbf{u}}\beta_c\int_0^{t_1}\|\nabla e_c^m\|_{\mathbf{L}^1(\Omega)}\,ds \\
&\quad + \beta_c\int_0^{t_1}\|\mathbf{e}_{\mathbf{u}}^{m-1}\|_{\mathbf{L}^2(\Omega)}\|\nabla e_c^m\|_{\mathbf{L}^2(\Omega)}\,ds + \beta_{\mathbf{u}}\|e_c^m(t_1)\|_{L^2(\Omega)}\|\nabla e_c^m(t_1)\|_{\mathbf{L}^2(\Omega)} \\
&\quad + \beta_{\mathbf{u}}\int_0^{t_1}\|e_c^m\|_{L^2(\Omega)}\|\nabla e_c^m\|_{\mathbf{L}^2(\Omega)}\,ds + \beta_{\mathbf{u}}\int_0^{t_1}\|\partial_t e_c^m\|_{L^2(\Omega)}\|\nabla e_c^m\|_{\mathbf{L}^2(\Omega)}\,ds \\
&\quad + \beta_c\|\mathbf{e}_{\mathbf{u}}^{m-1}(t_1)\|_{\mathbf{L}^2(\Omega)}\|\nabla e_c^m(t_1)\|_{\mathbf{L}^2(\Omega)} + \beta_c\int_0^{t_1}\|\mathbf{e}_{\mathbf{u}}^{m-1}\|_{\mathbf{L}^2(\Omega)}\|\nabla e_c^m\|_{\mathbf{L}^2(\Omega)}\,ds\Bigg)\,.
\end{aligned}$$

Using (2.13) and Young's inequality gives

$$\int_0^{t_1}\|\partial_t e_c^m\|^2_{L^2(\Omega)}\,ds + \|\nabla e_c^m(t_1)\|^2_{\mathbf{L}^2(\Omega)} \;\le\; C\bigg(\int_0^{t_1}\Big(\|\nabla e_c^m\|^2_{\mathbf{L}^2(\Omega)} + \|e_c^m\|^2_{L^2(\Omega)}\Big)\,ds$$

$$+\int_0^{t_1}\|\mathbf{e}_{\mathbf{u}}^{m-1}\|^2_{\mathbf{L}^2(\Omega)}\,ds+\|e_c^m(t_1)\|^2_{L^2(\Omega)}+\|\mathbf{e}_{\mathbf{u}}^{m-1}(t_1)\|^2_{\mathbf{L}^2(\Omega)}+\int_0^{t_1}\|\nabla e_c^m\|_{\mathbf{L}^1(\Omega)}\,ds\Bigg)\,. \tag{4.32}$$

Combining (4.32) with (4.17), (4.20), and (4.25), we conclude that

$$\|\nabla e_c^m(t_1)\|^2_{\mathbf{L}^2(\Omega)}+\int_0^{t_1}\|\partial_t e_c^m\|^2_{L^2(\Omega)}\,ds\to 0,\ \text{as } m\to\infty\,. \tag{4.33}$$

Combining (4.25)–(4.27) and (4.33) implies that, as $m\to\infty$,

$$\begin{aligned}(\mathbf{e}^m_{\mathbf{u}_f},\mathbf{e}^m_{\mathbf{u}_p},\mathbf{e}^m_{\boldsymbol{\eta}_p})\to\mathbf{0}\quad &\text{in}\quad L^\infty(0,t_1;\mathbf{V}_f)\times(L^\infty(0,t_1;L^2(\Omega_p))\cap L^2(0,t_1;\mathbf{V}_p))\times H^1(0,t_1;\mathbf{X}_p)\,;\\ (e^m_{p_f},e^m_{p_p},e^m_\lambda)\to\mathbf{0}\quad &\text{in}\quad L^\infty(0,t_1;W_f)\times H^1(0,t_1;W_p)\times L^\infty(0,t_1;\Lambda)\,;\\ e_c^m\to 0\quad &\text{in}\quad H^1(0,t_1;L^2(\Omega))\cap L^\infty(0,t_1;X_c)\,.\end{aligned} \tag{4.34}$$

**Step 5: Establish local in time existence of a solution.**

From (4.34) we conclude that $(\mathbf{u}_f^m,p_f^m,\boldsymbol{\eta}_p^m,\mathbf{u}_p^m,p_p^m,\lambda^m,c^m)\to(\mathbf{u}_f,p_f,\boldsymbol{\eta}_p,\mathbf{u}_p,p_p,\lambda,c)$ in the above norms as $m\to\infty$. We next take $m\to\infty$ in (**LP1m**) and (**LP2m**) to conclude that the limit functions solve the weak formulation (**P**) on $(0,t_1]$. Indeed, the convergence of all linear terms is immediate. Below we discuss the convergence of the nonlinear terms. For the viscosity terms $(\mu(c^{m-1})K^{-1}\mathbf{u}_p^m,\mathbf{v}_p)_{\Omega_p}$ and $(2\mu(c^{m-1})\boldsymbol{\epsilon}(\mathbf{u}_f^m),\boldsymbol{\epsilon}(\mathbf{v}_f))_{\Omega_f}$, since $c^m\to c$ in $L^\infty(0,T;L^2(\Omega))$, we conclude that $c^m\to c$ a.e. in $\Omega\times(0,t_1)$ as $m\to\infty$. For any $\mathbf{v}_p\in\mathbf{L}^2(\Omega_p)$ and $\mathbf{v}_f\in\mathbf{H}^1(\Omega_f)$, since $\mu(c)$ is Lipschitz, we have, as $m\to\infty$,

$$\mu(c^m)\mathbf{v}_p\to\mu(c)\mathbf{v}_p\ \text{ a.e. in } \Omega_p\times(0,t_1),\quad \mu(c^m)\boldsymbol{\epsilon}(\mathbf{v}_f)\to\mu(c)\boldsymbol{\epsilon}(\mathbf{v}_f)\ \text{ a.e. in } \Omega_f\times(0,t_1)\,.$$

In addition,

$$|\mu(c^m)\mathbf{v}_p|\ \le\ \mu_U|\mathbf{v}_p|\ \text{ a.e. in } \Omega_p\times(0,t_1),\quad |\mu(c^m)\boldsymbol{\epsilon}(\mathbf{v}_f)|\le\mu_U|\boldsymbol{\epsilon}(\mathbf{v}_f)|\ \text{ a.e. in } \Omega_f\times(0,t_1)\,.$$

Therefore, by the Lebesgue dominated convergence theorem [47, Page 88], we have as $m\to\infty$,

$$\mu(c^m)\mathbf{v}_p\to\mu(c)\mathbf{v}_p \text{ in } \mathbf{L}^2(\Omega_p),\quad \mu(c^m)\boldsymbol{\epsilon}(\mathbf{v}_f)\to\mu(c)\boldsymbol{\epsilon}(\mathbf{v}_f) \text{ in } \mathbf{L}^2(\Omega_f)\ \text{ a.e. in } (0,t_1)\,.$$

In turn, from (4.34) we know that as $m\to\infty$, $\mathbf{u}_p^m\to\mathbf{u}_p$ in $\mathbf{L}^2(\Omega_p)$ and $\mathbf{u}_f^m\to\mathbf{u}_f$ in $\mathbf{H}^1(\Omega_f)$ a.e. in $(0,t_1)$. Therefore, as $m\to\infty$, a.e. in $(0,t_1)$,

$$(\mu(c^{m-1})K^{-1}\mathbf{u}_p^m,\mathbf{v}_p)_{\Omega_p}\to(\mu(c)K^{-1}\mathbf{u}_p,\mathbf{v}_p)_{\Omega_p},\ (2\mu(c^{m-1})\boldsymbol{\epsilon}(\mathbf{u}_f^m),\boldsymbol{\epsilon}(\mathbf{v}_f))_{\Omega_f}\to(2\mu(c)\boldsymbol{\epsilon}(\mathbf{u}_f),\boldsymbol{\epsilon}(\mathbf{v}_f))_{\Omega_f}.$$

Similarly, for the transport subproblem (**LP2m**), we have for any $\psi\in X_c$, a.e. in $(0,t_1)$,

$$(\mathbf{D}(\mathbf{u}^{m-1})\nabla c^m,\nabla\psi)\to(\mathbf{D}(\mathbf{u})\nabla c,\nabla\psi)\quad\text{and}\quad(c^m\mathbf{u}^{m-1},\nabla\psi)\to(c\mathbf{u},\nabla\psi)\quad\text{as } m\to\infty\,,$$

using $|\mathbf{D}(\mathbf{u}^{m-1})\nabla\psi|\le\phi^*(d_m+(\alpha_l+\alpha_t)\beta_{\mathbf{u}})|\nabla\psi|$ and the Lebesgue dominated convergence theorem.

We finally note that, since all functions lie in $L^\infty(0,t_1)$, cf. (4.34), we can take $m\to\infty$ in $(\mathbf{u}_f^m(0),p_f^m(0),\boldsymbol{\eta}_p^m(0),\mathbf{u}_p^m(0),p_p^m(0),\lambda^m(0),c^m(0))=(\mathbf{u}_{f,0},p_{f,0},\boldsymbol{\eta}_{p,0},\mathbf{u}_{p,0},p_{p,0},\lambda_0,c_0)$, cf. Remark 4.1, to conclude that $(\mathbf{u}_f(0),p_f(0),\boldsymbol{\eta}_p(0),\mathbf{u}_p(0),p_p(0),\lambda(0),c(0))=(\mathbf{u}_{f,0},p_{f,0},\boldsymbol{\eta}_{p,0},\mathbf{u}_{p,0},p_{p,0},\lambda_0,c_0)$.

**Step 6: Extend the local in time result to $(0,T]$.**

Let $[0,T]=\sum_{k=1}^N[t_{k-1},t_k]$ satisfying $t_0=0$, $t_N=T$ and $\sup_{1\le k\le N}|t_k-t_{k-1}|\le\frac{1}{2L}$, where $L=C\exp(T)$ from (4.16). Then, for $k=2,\ldots,N$, we consider the iteration based on (**LP1m**) and (**LP2m**) on the interval $[t_{k-1},t_k]$ with initial data $\boldsymbol{\eta}_p(t_{k-1})$, $p_p(t_{k-1})$, and $c(t_{k-1})$, the solution of (**P**) at $t=t_{k-1}$ obtained in Step 4. The argument from Steps 1–4 implies that the sequence $(\mathbf{u}_f^m,p_f^m,\boldsymbol{\eta}_p^m,\mathbf{u}_p^m,p_p^m,\lambda^m,c^m)$ converges to a solution of (**P**) on $[t_{k-1},t_k]$ with all variables being continuous at $t_{k-1}$ for $k=2,\ldots,N$, which establishes existence of a solution to (**P**) on $[0,T]$.

**Step 7: Uniqueness of the solution.**

The convergence result (4.34) implies that $\mathbf{u}_f^m \to \mathbf{u}_f$ in $L^\infty(0,T;\mathbf{V}_f)$, $\mathbf{u}_p^m \to \mathbf{u}_p$ in $L^\infty(0,T;L^2(\Omega_p))$, and $c^m \to c$ in $L^\infty(0,T;X_c)$. Therefore, $\mathbf{u}_f^m \to \mathbf{u}_f$ and $D\,\mathbf{u}_f^m \to D\,\mathbf{u}_f$ a.e. in $\Omega_f \times (0,T)$, where $D$ denotes any first order weak derivative. Similarly, $\mathbf{u}_p^m \to \mathbf{u}_p$ a.e. in $\Omega_p \times (0,T)$, and $c^m \to c$, $D\,c^m \to D\,c$ a.e. in $\Omega \times (0,T)$. Combined with Assumption 4.1, we have that

$$\max\left\{\|\mathbf{u}_f\|_{L^\infty(0,T;\mathbf{W}^{1,\infty}(\Omega_f))},\, \|\mathbf{u}_p\|_{L^\infty(0,T;\mathbf{L}^\infty(\Omega_p))}\right\} \le \beta_{\mathbf{u}}\,, \quad \|c\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} \le \beta_c\,. \tag{4.35}$$

We note that the time regularity in (4.35) is $L^\infty(0,T)$, which is lower than the one in Assumption 4.1. This is sufficient for the uniqueness argument, which does not involve differentiation in time.

Assume that there exists another solution $(\tilde{\mathbf{u}}_f, \tilde{p}_f, \tilde{\boldsymbol{\eta}}_p, \tilde{\mathbf{u}}_p, \tilde{p}_p, \tilde{\lambda}, \tilde{c})$ of problem $(\mathbf{P})$ with initial data $\tilde{p}_p(0) = p_{p,0}$, $\tilde{\boldsymbol{\eta}}_p(0) = \boldsymbol{\eta}_{p,0}$, and $\tilde{c}(0) = c_0$. Denote by $(\mathbf{e}_{\mathbf{u}_f}, e_{p_f}, \mathbf{e}_{\boldsymbol{\eta}_p}, \mathbf{e}_{\mathbf{u}_p}, e_{p_p}, e_\lambda, e_c)$ the difference of the two solutions. Subtracting the systems (2.11) for the two solutions results in $(\mathbf{e}_{\mathbf{u}_f}, e_{p_f}, \mathbf{e}_{\boldsymbol{\eta}_p}, \mathbf{e}_{\mathbf{u}_p}, e_{p_p}, e_\lambda, e_c) : [0,T] \to \mathbf{V}_f \times W_f \times \mathbf{X}_p \times \mathbf{V}_p \times W_p \times \Lambda \times X_c$ such that for all $(\mathbf{v}_f, w_f, \boldsymbol{\xi}_p, \mathbf{v}_p, w_p, \nu, \psi) \in \mathbf{V}_f \times W_f \times \mathbf{X}_p \times \mathbf{V}_p \times W_p \times \Lambda \times X_c$ and for a.e. $t \in (0,T)$,

$$\begin{aligned}
& a_f(\mathbf{e}_{\mathbf{u}_f}, \mathbf{v}_f; c) + \big(2(\mu(c) - \mu(\tilde{c}))\boldsymbol{\epsilon}(\mathbf{u}_f), \boldsymbol{\epsilon}(\mathbf{v}_f)\big)_{\Omega_f} + a_{BJS}(\mathbf{e}_{\mathbf{u}_f}, \partial_t \mathbf{e}_{\boldsymbol{\eta}_p}; \mathbf{v}_f, 0) \\
& \quad + b_f(\mathbf{v}_f, e_{p_f}) + b_\Gamma(\mathbf{v}_f, 0, 0; e_\lambda) = 0\,, \\
& -b_f(\mathbf{e}_{\mathbf{u}_f}, w_f) = 0\,, \\
& a_p^e(\mathbf{e}_{\boldsymbol{\eta}_p}, \boldsymbol{\xi}_p) + \alpha b_e(\boldsymbol{\xi}_p, e_{p_p}) + a_{BJS}(\mathbf{e}_{\mathbf{u}_f}, \partial_t \mathbf{e}_{\boldsymbol{\eta}_p}; 0, \boldsymbol{\xi}_p) + b_\Gamma(0, 0, \boldsymbol{\xi}_p; e_\lambda) = 0\,, \\
& a_p^d(\mathbf{e}_{\mathbf{u}_p}, \mathbf{v}_p; c) + \big((\mu(c) - \mu(\tilde{c}))K^{-1}\mathbf{u}_p, \mathbf{v}_p\big)_{\Omega_p} + b_p(\mathbf{v}_p, e_{p_p}) + b_\Gamma(0, \mathbf{v}_p, 0; e_\lambda) = 0\,, \\
& s_0(\partial_t e_{p_p}, w_p)_{\Omega_p} - \alpha b_e(\partial_t \mathbf{e}_{\boldsymbol{\eta}_p}, w_p) - b_p(\mathbf{e}_{\mathbf{u}_p}, w_p) = 0\,, \\
& b_\Gamma(\mathbf{e}_{\mathbf{u}_f}, \mathbf{e}_{\mathbf{u}_p}, \partial_t \mathbf{e}_{\boldsymbol{\eta}_p}; \nu) = 0\,, \\
& (\phi \partial_t e_c, \psi) + (\mathbf{D}(\mathbf{u})\nabla e_c, \nabla\psi) + ((\mathbf{D}(\mathbf{u}) - \mathbf{D}(\tilde{\mathbf{u}}))\nabla c, \nabla\psi) - (e_c \mathbf{u}, \nabla\psi) - (c\,\mathbf{e}_{\mathbf{u}}, \nabla\psi) - (q^- e_c, \psi) = 0\,,
\end{aligned}$$

with $e_{p_p}(0) = 0$, $\mathbf{e}_{\boldsymbol{\eta}_p}(0) = \mathbf{0}$, and $e_c(0) = 0$. We note the similarity of the above system with (4.3). The arguments from Steps 1–3 leading to (4.16) imply that

$$\int_0^{t_1} \|e_c\|^2_{L^2(\Omega)}\, ds \le \frac{1}{2}\int_0^{t_1} \|e_c\|^2_{L^2(\Omega)}\, ds\,, \tag{4.36}$$

and hence $e_c = 0$ on $(0,t_1)$, i.e., $c$ is unique on $(0,t_1)$. We emphasize that in the bounds corresponding to (4.5), (4.6), (4.8), (4.12), and (4.13) we utilize the extra regularity (4.35) for the solution obtained in Steps 1-6, while no such regularity is assumed for the other solution. Next, using the arguments from Step 1 leading to (4.10) we obtain

$$\begin{aligned}
& \|e_{p_p}(t_1)\|^2_{L^2(\Omega_p)} + \|\mathbf{e}_{\boldsymbol{\eta}_p}(t_1)\|^2_{\mathbf{H}^1(\Omega_p)} + \int_0^{t_1} \Big(\|\mathbf{e}_{\mathbf{u}_f}\|^2_{\mathbf{H}^1(\Omega_f)} + \|\mathbf{e}_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} + \|e_{p_f}\|^2_{L^2(\Omega_f)} \\
& \quad + \|e_{p_p}\|^2_{L^2(\Omega_p)} + \|e_\lambda\|^2_\Lambda + |\mathbf{e}_{\mathbf{u}_f} - \partial_t \mathbf{e}_{\boldsymbol{\eta}_p}|^2_{BJS}\Big)\, ds \,\le\, C\int_0^{t_1} \|e_c\|^2_{L^2(\Omega)}\, ds\,,
\end{aligned} \tag{4.37}$$

concluding that the solution $(\mathbf{u}_f, p_f, \boldsymbol{\eta}_p, \mathbf{u}_p, p_p, \lambda)$ of the Stokes–Biot subproblem is unique on the interval $(0,t_1)$. The final step is to extend the uniqueness result globally in time to $(0,T)$, using the process from Step 6. $\square$

## 5 Semi-discrete numerical approximation of problem (P)

In this section we introduce the semi-discrete continuous-in-time numerical approximation of the fully coupled problem (**P**), cf. (2.11) and present its well-posedness analysis. The proof follows the approach developed in the previous sections for the solvability of problem (**P**).

Recall that in Section 3.1 we considered the semi-discrete Galerkin approximations (**LGP1**) and (**LGP2**) of the linearized and decoupled Stokes-Biot and transport problems (**LP1**) and (**LP2**). There we introduced the finite element partition $\mathcal{T}_h = \mathcal{T}_h^f \cup \mathcal{T}_h^p$, the Stokes finite element spaces $\mathbf{V}_{fh} \times W_{fh} \subset \mathbf{V}_f \times W_f$, the Darcy finite spaces $\mathbf{V}_{ph} \times W_{ph} \subset \mathbf{V}_p \times W_p$, the displacement finite element space $\mathbf{X}_{ph} \subset \mathbf{X}_p$ and the concentration finite element space $X_{ch} \subset X_c$. We utilize the same spaces here. In Section 3.1 we also used the conforming Lagrange multiplier finite element space $\Lambda_h \subset \Lambda$, which allowed us in Section 3.4 to pass to the limit $h \to 0$ in the Galerkin problems. Such step is not needed here, since we are already in a semi-discrete setting. In addition, the choice $\Lambda_h \subset \Lambda$ leads to a suboptimal interface error term, cf. Remark 6.1. Instead, we utilize the non-conforming discrete Lagrange multiplier space

$$\tilde{\Lambda}_h = \mathbf{V}_{ph} \cdot \mathbf{n}_p|_{\Gamma_{fp}}\,, \tag{5.1}$$

which consists of discontinuous piecewise polynomials. We equip $\tilde{\Lambda}_h$ with the norm $\|\nu_h\|^2_{\tilde{\Lambda}_h} = \|\nu_h\|^2_{L^2(\Gamma_{fp})}$. Due to this change we need a new version of the discrete inf-sup condition, which is shown in [1, Theorem 5.2]. There exists a constant $\beta_3 > 0$ such that for all $(w_{fh}, w_{ph}, \nu_h) \in W_{fh} \times W_{ph} \times \tilde{\Lambda}_h$, there holds

$$\sup_{\mathbf{0} \neq (\mathbf{v}_{fh}, \mathbf{v}_{ph}) \in \mathbf{V}_f \times \mathbf{V}_p} \frac{b_f(\mathbf{v}_{fh}, w_{fh}) + b_p(\mathbf{v}_{ph}, w_{ph}) + b_\Gamma(\mathbf{v}_{fh}, \mathbf{v}_{ph}, \mathbf{0}; \nu_h)}{\|(\mathbf{v}_{fh}, \mathbf{v}_{ph})\|_{\mathbf{V}_f \times \mathbf{V}_p}} \geq \beta_3 \|(w_{fh}, w_{ph}, \nu_h)\|_{W_f \times W_p \times \tilde{\Lambda}_h}. \tag{5.2}$$

**Remark 5.1** *The above inf-sup condition holds for non-matching grids along the interface, hence the analysis of the numerical method also holds for non-matching grids.*

We consider the following semi-discrete approximation of (**P**):

(**DP**): Find $(\mathbf{u}_{fh}, p_{fh}, \boldsymbol{\eta}_{ph}, \mathbf{u}_{ph}, p_{ph}, \lambda_h, c_h) : [0,T] \to \mathbf{V}_{fh} \times W_{fh} \times \mathbf{X}_{ph} \times \mathbf{V}_{ph} \times W_{ph} \times \tilde{\Lambda}_h \times X_{ch}$ such that for a.e. $t \in (0,T)$ and for all $(\mathbf{v}_{fh}, w_{fh}, \boldsymbol{\xi}_{ph}, \mathbf{v}_{ph}, w_{ph}, \nu_h, \psi_h) \in \mathbf{V}_{fh} \times W_{fh} \times \mathbf{X}_{ph} \times \mathbf{V}_{ph} \times W_{ph} \times \tilde{\Lambda}_h \times X_{ch}$,

$$a_f(\mathbf{u}_{fh}, \mathbf{v}_{fh}; c_h) + b_f(\mathbf{v}_{fh}, p_{fh}) + a_{BJS}(\mathbf{u}_{fh}, \partial_t \boldsymbol{\eta}_{ph}; \mathbf{v}_{fh}, 0) + b_\Gamma(\mathbf{v}_{fh}, 0, 0; \lambda_h) = (\mathbf{f}_f, \mathbf{v}_{fh})_{\Omega_f}\,, \tag{5.3a}$$

$$- b_f(\mathbf{u}_{fh}, w_{fh}) = (q_f, w_{fh})_{\Omega_f}\,, \tag{5.3b}$$

$$a_p^e(\boldsymbol{\eta}_{ph}, \boldsymbol{\xi}_{ph}) + \alpha b_e(\boldsymbol{\xi}_{ph}, p_{ph}) + a_{BJS}(\mathbf{u}_{fh}, \partial_t \boldsymbol{\eta}_{ph}; 0, \boldsymbol{\xi}_{ph}) + b_\Gamma(0, 0, \boldsymbol{\xi}_{ph}; \lambda_h) = (\mathbf{f}_p, \boldsymbol{\xi}_{ph})_{\Omega_p}\,, \tag{5.3c}$$

$$a_p^d(\mathbf{u}_{ph}, \mathbf{v}_{ph}; c_h) + b_p(\mathbf{v}_{ph}, p_{ph}) + b_\Gamma(0, \mathbf{v}_{ph}, 0; \lambda_h) = 0\,, \tag{5.3d}$$

$$s_0(\partial_t p_{ph}, w_{ph})_{\Omega_p} - b_p(\mathbf{u}_{ph}, w_{ph}) - \alpha b_e(\partial_t \boldsymbol{\eta}_{ph}, w_{ph}) = (q_p, w_{ph})_{\Omega_p}\,, \tag{5.3e}$$

$$b_\Gamma(\mathbf{u}_{fh}, \mathbf{u}_{ph}, \partial_t \boldsymbol{\eta}_{ph}; \nu_h) = 0\,, \tag{5.3f}$$

$$(\phi \partial_t c_h, \psi_h) + (\mathbf{D}(\mathbf{u}_h) \nabla c_h, \nabla \psi_h) - (c_h \mathbf{u}_h, \nabla \psi_h) - (q^- c_h, \psi_h) = (q^+ c_w, \psi_h)\,, \tag{5.3g}$$

The well-posedness and stability analysis of (**DP**) follows closely the analysis of (**P**), using the following steps:

- Consider linearized and decoupled subproblems (**LDP1**) and (**LDP2**), establish their well-posedness, and develop stability bounds.
- Use a fixed point iteration argument to establish well-posedness of (**DP**).

### 5.1 Linearized and decoupled subproblems (LDP1) and (LDP2)

The Stokes-Biot subproblem (**LDP1**) is as defined as follows.

(**LDP1**): Given $\mathbf{f}_f$, $\mathbf{f}_p$, $q_p$, $q_f$, and $\gamma_h : [0,T] \to X_{ch}$, find $(\mathbf{u}_{fh}, p_{fh}, \boldsymbol{\eta}_{ph}, \mathbf{u}_{ph}, p_{ph}, \lambda_h) : [0,T] \to \mathbf{V}_{fh} \times W_{fh} \times \mathbf{X}_{ph} \times \mathbf{V}_{ph} \times W_{ph} \times \tilde{\Lambda}_h$ such that for a.e. $t \in (0,T)$ and for all $(\mathbf{v}_{fh}, w_{fh}, \boldsymbol{\xi}_{ph}, \mathbf{v}_{ph}, w_{ph}, \nu_h) \in \mathbf{V}_{fh} \times W_{fh} \times \mathbf{X}_{ph} \times \mathbf{V}_{ph} \times W_{ph} \times \tilde{\Lambda}_h$,

$$a_f(\mathbf{u}_{fh}, \mathbf{v}_{fh}; \gamma_h) + b_f(\mathbf{v}_{fh}, p_{fh}) + a_{BJS}(\mathbf{u}_{fh}, \partial_t \boldsymbol{\eta}_{ph}; \mathbf{v}_{fh}, 0) + b_\Gamma(\mathbf{v}_{fh}, 0, 0; \lambda_h) = (\mathbf{f}_f, \mathbf{v}_{fh})_{\Omega_f}\,, \tag{5.4a}$$

$$- b_f(\mathbf{u}_{fh}, w_{fh}) = (q_f, w_{fh})_{\Omega_f}\,, \tag{5.4b}$$

$$a_p^e(\boldsymbol{\eta}_{ph}, \boldsymbol{\xi}_{ph}) + \alpha b_e(\boldsymbol{\xi}_{ph}, p_{ph}) + a_{BJS}(\mathbf{u}_{fh}, \partial_t \boldsymbol{\eta}_{ph}; 0, \boldsymbol{\xi}_{ph}) + b_\Gamma(0, 0, \boldsymbol{\xi}_{ph}; \lambda_h) = (\mathbf{f}_p, \boldsymbol{\xi}_{ph})_{\Omega_p}\,, \tag{5.4c}$$

$$a_p^d(\mathbf{u}_{ph}, \mathbf{v}_{ph}; \gamma_h) + b_p(\mathbf{v}_{ph}, p_{ph}) + b_\Gamma(0, \mathbf{v}_{ph}, 0; \lambda_h) = 0\,, \tag{5.4d}$$

$$s_0(\partial_t p_{ph}, w_{ph})_{\Omega_p} - b_p(\mathbf{u}_{ph}, w_{ph}) - \alpha b_e(\partial_t \boldsymbol{\eta}_{ph}, w_{ph}) = (q_p, w_{ph})_{\Omega_p}\,, \tag{5.4e}$$

$$b_\Gamma(\mathbf{u}_{fh}, \mathbf{u}_{ph}, \partial_t \boldsymbol{\eta}_{ph}; \nu_h) = 0\,, \tag{5.4f}$$

The transport subproblem (**LDP2**) is defined as follows.

(**LDP2**): Given $q$, $c_w$, and $\boldsymbol{\theta}_h : [0,T] \to \mathbf{V}_h$, find $c_h : [0,T] \to X_{ch}$ such that for all $\psi_h \in X_{ch}$ and for a.e. $t \in (0,T)$,

$$(\phi \partial_t c_h, \psi_h) + (\mathbf{D}(\boldsymbol{\theta}_h)\nabla c_h, \nabla \psi_h) - (c_h \boldsymbol{\theta}_h, \nabla \psi_h) - (q^- c_h, \psi_h) = (q^+ c_w, \psi_h)\,. \tag{5.5}$$

**Remark 5.2** *The only difference between* (**LDP1**) *and* (**LGP1**)*, which was used in the well-posedness of the continuous problem, is that the concentration data $\gamma_h$ in* (**LDP1**) *is discrete. In particular, the two non-linear terms $a_f$ in (5.4a) and $a_p^d$ in (5.4c) involve viscosity $\mu(\gamma_h)$. Similarly, the difference between* (**LDP1**) *and* (**LGP1**) *is that the velocity data $\boldsymbol{\theta}_h$ is discrete.*

Using the discrete inf-sup condition (5.2), the proof of the following theorem on the solvability and stability of (**LDP1**) and (**LDP2**) is similar to the proofs of Theorems 3.2 and 3.4.

**Theorem 5.1** *Assume that $\mathbf{f}_f \in L^\infty(0,T;\mathbf{V}_f')$, $\mathbf{f}_p \in H^1(0,T;\mathbf{X}_p')$, $q_f \in L^\infty(0,T;W_f')$, $q_p \in L^\infty(0,T;W_p')$, $\gamma_h \in L^\infty(0,T;L^\infty(\Omega))$, and $p_{p,0} \in H_p$. Let the initial data $(\mathbf{u}_{fh,0}, p_{fh,0}, \boldsymbol{\eta}_{ph,0}, \mathbf{u}_{ph,0}, p_{ph,0}, \lambda_{h,0}) \in \mathbf{V}_{fh} \times W_{fh} \times \mathbf{X}_{ph} \times \mathbf{V}_{ph} \times W_{ph} \times \tilde{\Lambda}_h$ be constructed in (3.18) and (3.19). Then problem* (**LDP1**) *with initial conditions $p_{ph}(0) = p_{ph,0}$ and $\boldsymbol{\eta}_{ph}(0) = \boldsymbol{\eta}_{ph,0}$ has a unique solution $(\mathbf{u}_{fh}, \mathbf{u}_{ph}, \boldsymbol{\eta}_{ph}, p_{fh}, p_{ph}, \lambda_h)$. Moreover, the solution satisfies $\mathbf{u}_{fh}(0) = \mathbf{u}_{fh,0}$, $p_{fh}(0) = p_{fh,0}$, $\mathbf{u}_{ph}(0) = \mathbf{u}_{ph,0}$, and $\lambda_h(0) = \lambda_{h,0}$. There exists a positive constant $C$ independent of $h$ such that*

$$\begin{aligned}
&\sqrt{s_0}\, \|p_{ph}\|_{L^\infty(0,T;W_p)} + \|\boldsymbol{\eta}_{ph}\|_{L^\infty(0,T;\mathbf{X}_p)} + \|\mathbf{u}_{fh}\|_{L^2(0,T;\mathbf{V}_f)} + \|p_{fh}\|_{L^2(0,T;W_f)} \\
&\quad + \|\mathbf{u}_{ph}\|_{L^2(0,T;\mathbf{L}^2(\Omega_p))} + \|p_{ph}\|_{L^2(0,T;W_p)} + \|\lambda_h\|_{L^2(0,T;\tilde{\Lambda}_h)} + |\mathbf{u}_{fh} - \partial_t \boldsymbol{\eta}_{ph}|_{L^2(0,T;BJS)} \\
&\le C\Big( \|\mathbf{f}_f\|_{L^2(0,T;\mathbf{V}_f')} + \|\mathbf{f}_p\|_{L^2(0,T;\mathbf{X}_p')} + \|\partial_t \mathbf{f}_p\|_{L^1(0,T;\mathbf{X}_p')} + \|q_p\|_{L^2(0,T;W_p')} \\
&\quad + \|q_f\|_{L^2(0,T;W_f')} + \|p_{p,0}\|_{H^1(\Omega_p)} + \|\mu(c_0)^{-1} K \nabla p_{p,0}\|_{\mathbf{H}^1(\Omega_p)} \Big)\,.
\end{aligned} \tag{5.6}$$

*Moreover, if* $\mathbf{f}_f \in H^1(0,T;\mathbf{V}_f')$, $\mathbf{f}_p \in H^2(0,T;\mathbf{X}_p')$, $q_f \in H^1(0,T;W_f')$, $q_p \in H^1(0,T;W_p')$, *and* $\gamma_h \in W^{1,\infty}(0,T;L^\infty(\Omega))$, *it holds that*

$$\begin{aligned}
&\sqrt{s_0}\|\partial_t p_{ph}\|_{L^\infty(0,T;W_p)} + \|\partial_t p_{ph}\|_{L^2(0,T;W_p)} + \|\partial_t p_{fh}\|_{L^2(0,T;W_f)} + \|\partial_t \boldsymbol{\eta}_{ph}\|_{L^\infty(0,T;\mathbf{X}_p)} \\
&\quad + \|\partial_t \lambda_h\|_{L^2(0,T;\tilde{\Lambda}_h)} + \|\partial_t \mathbf{u}_{fh}\|_{L^2(0,T;\mathbf{V}_f)} + \|\nabla\cdot\mathbf{u}_{ph}\|_{L^2(0,T;L^2(\Omega_p))} \\
&\quad + \|\partial_t \mathbf{u}_{ph}\|_{L^2(0,T;\mathbf{L}^2(\Omega_p))} + |\mathbf{u}_{fh} - \partial_t\boldsymbol{\eta}_{ph}|_{L^\infty(0,T;BJS)} \\
&\le C\Big(\|\mathbf{f}_f\|_{H^1(0,T;\mathbf{V}_f')} + \|\mathbf{f}_p\|_{H^1(0,T;\mathbf{X}_p')} + \|\partial_{tt}\mathbf{f}_p\|_{L^1(0,T;\mathbf{X}_p')} + \|q_f\|_{H^1(0,T;W_f')} + \|q_p\|_{H^1(0,T;W_p')} \\
&\quad + \frac{1}{\sqrt{s_0}}\big(\|p_{p,0}\|_{H^1(\Omega_p)} + \big\|\mu(c_0)^{-1}K\nabla p_{p,0}\big\|_{\mathbf{H}^1(\Omega_p)} + \|\mathbf{f}_f(0)\|_{\mathbf{V}_f'} + \|q_f(0)\|_{W_f'} + \|q_p(0)\|_{W_p'}\big)\Big). \quad (5.7)
\end{aligned}$$

*Let* $q^- \in L^\infty(0,T;L^\infty(\Omega))$, $q^+ c_w \in L^\infty(0,T;X_c')$, $\boldsymbol{\theta}_h \in L^\infty(0,T;\mathbf{L}^\infty(\Omega))$, *and* $c_0 \in H^1(\Omega)$. *Then,* (**LDP2**) *has a unique solution* $c_h$ *satisfying* $c_h(0) = c_{h,0}$, *where* $c_{h,0}$ *is defined in* (3.9). *It holds that*

$$\|c_h\|_{L^\infty(0,T;L^2(\Omega))} + \|c_h\|_{L^2(0,T;X_c)} \le C\sqrt{\exp(LT)}\left(\|c_0\|_{H^1(\Omega)} + \|q^+c_w\|_{L^2(0,T;X_c')}\right), \quad (5.8)$$

*where* $L = \|\phi^{-1/2}\mathbf{D}(\boldsymbol{\theta}_h)^{-1/2}\boldsymbol{\theta}_h\|^2_{L^\infty(0,T;\mathbf{L}^\infty(\Omega))}$. *Furthermore, if* $q^+c_w \in L^\infty(0,T;L^2(\Omega))$ *and* $\boldsymbol{\theta}_h \in W^{1,\infty}(0,T;\mathbf{L}^\infty(\Omega))$, *it holds that*

$$\|\partial_t c_h\|_{L^2(0,T;L^2(\Omega))} + \|c_h\|_{L^\infty(0,T;X_c)} \le C\sqrt{\exp(LT)}\left(\|c_0\|_{H^1(\Omega)} + \|q^+c_w\|_{L^2(0,T;L^2(\Omega))}\right). \quad (5.9)$$

### 5.2 Well-posedness of the coupled semi-discrete problem (DP)

Similarly to the continuous case, we consider an iteration of decoupled linearized problems (**LDP1**) and (**LDP2**) and utilize a fixed point iteration argument to establish convergence. In particular, for $m \ge 1$, we solve a Stokes-Biot problem (**LDP1m**) with concentration data $c_h^{m-1}$ and a transport problem (**LDP2m**) with velocity data $\mathbf{u}_h^{m-1}$. Similarly to Assumption 4.1, we make the following assumption on the regularity of $\mathbf{u}_h^m$ and $c_h^m$.

**Assumption 5.1** *For all* $m \ge 0$, $\mathbf{u}_{fh}^m \in W^{1,\infty}(0,T;\mathbf{W}^{1,\infty}(\Omega_f))$, $\mathbf{u}_{ph}^m \in W^{1,\infty}(0,T;\mathbf{L}^\infty(\Omega_p))$, *and* $c_h^m \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega))$, *with finite constants* $\tilde{\beta}_{\mathbf{u}}$ *and* $\tilde{\beta}_c$, *where*

$$\tilde{\beta}_{\mathbf{u}} := \max\Big\{\sup_m \|\mathbf{u}_{fh}^m\|_{W^{1,\infty}(0,T;\mathbf{W}^{1,\infty}(\Omega_f))}, \sup_m \|\mathbf{u}_{ph}^m\|_{W^{1,\infty}(0,T;\mathbf{L}^\infty(\Omega_p))}\Big\}$$

$$\text{and} \quad \tilde{\beta}_c := \sup_m \|c_h^m\|_{W^{1,\infty}(0,T;W^{1,\infty}(\Omega))}.$$

Assumption 5.1 can be justified using Assumption 4.1 for the continuous versions of (**LDP1m**) and (**LDP2m**), together with estimates of the numerical error in (**LDP1m**) and (**LDP2m**). We refer the reader to [2, Lemma 4.2] for details.

The proof of the following theorem on the well-posedness of the semi-discrete scheme (**DP**) follows the argument in the proof of Theorem 4.1.

**Theorem 5.2** *For each*

$$\mathbf{f}_f \in H^1(0,T;\mathbf{V}_f'), \quad \mathbf{f}_p \in H^1(0,T;\mathbf{V}_p'), \quad q_f \in H^1(0,T;W_f'), \quad q_p \in H^1(0,T;W_p'),$$

$$q^+c_w \in L^2(0,T;X_c'), \quad q^- \in H^1(0,T;L^\infty(\Omega)), \quad p_{p,0} \in H_p, \quad \text{and} \quad c_0 \in H^1(\Omega),$$

*and under Assumption 5.1, the semi-discrete scheme* ($\mathbf{DP}$) *has a unique solution* $(\mathbf{u}_{fh}, \mathbf{u}_{ph}, \boldsymbol{\eta}_{ph}, p_{fh}, p_{ph}, \lambda_h, c_h)$ *satisfying* $p_{ph}(0) = p_{ph,0}$, $\boldsymbol{\eta}_{ph}(0) = \boldsymbol{\eta}_{ph,0}$, *and* $c_h(0) = c_{h,0}$. *Moreover,* $\mathbf{u}_{fh}(0) = \mathbf{u}_{fh,0}$, $p_{fh}(0) = p_{fh,0}$, $\mathbf{u}_{ph}(0) = \mathbf{u}_{ph,0}$, $\lambda_h(0) = \lambda_{h,0}$, *and*

$$
\begin{aligned}
&(\mathbf{u}_{fh}, \mathbf{u}_{ph}, \boldsymbol{\eta}_{ph}) \in L^\infty(0,T;\mathbf{V}_{fh}) \times (L^\infty(0,T;\mathbf{L}^2(\Omega_p)) \cap L^2(0,T;\mathbf{V}_{ph})) \times H^1(0,T;\mathbf{X}_{ph})\,;\\
&(p_{fh}, p_{ph}, \lambda_h) \in L^\infty(0,T;W_{fh}) \times H^1(0,T;W_{ph}) \times L^\infty(0,T;\tilde{\Lambda}_h)\,;\\
&c_h \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;X_{ch})\,.
\end{aligned}
$$

Note that in the proof we obtain a bound on the numerical solution similar to (4.35):

$$
\max\left\{\|\mathbf{u}_{fh}\|_{L^\infty(0,T;\mathbf{W}^{1,\infty}(\Omega_f))},\, \|\mathbf{u}_{ph}\|_{L^\infty(0,T;\mathbf{L}^\infty(\Omega_p))}\right\} \le \tilde{\beta}_{\mathbf{u}}, \quad \|c_h\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} \le \tilde{\beta}_c. \tag{5.10}
$$

# 6 Error analysis

In this section, we analyze the spatial discretization error. For $\star \in \{f,p\}$, we denote by $k_\star$ and $s_\star$ the degrees of polynomials in the spaces $\mathbf{V}_{\star h}$ and $W_{\star h}$, respectively. Let $k_s$ and $k_c$ be the polynomial degree in $\mathbf{X}_{ph}$ and $X_{ch}$. We also recall the definition (5.1) of the non-conforming Lagrange multiplier finite element space $\tilde{\Lambda}_h$.

## 6.1 Approximation properties

Let $Q_{\star h}$ and $Q_{\lambda h}$ be the $L^2$-projection operators onto $W_{\star h}$ and $\tilde{\Lambda}_h$, respectively, satisfying:

$$
(p_\star - Q_{\star h} p_\star, w_{\star h})_{\Omega_\star} = 0 \quad \forall\, w_{\star h} \in W_{\star h}, \tag{6.1}
$$

$$
\langle \lambda - Q_{\lambda h}\lambda, \mu_h \rangle_{\Gamma_{fp}} = 0 \quad \forall\, \mu_h \in \tilde{\Lambda}_h. \tag{6.2}
$$

Note that, since $\tilde{\Lambda}_h = \mathbf{V}_{ph} \cdot \mathbf{n}_p|_{\Gamma_{fp}}$, (6.2) implies

$$
\langle \lambda - Q_{\lambda h}\lambda, \mathbf{v}_{ph} \cdot \mathbf{n}_p \rangle_{\Gamma_{fp}} = 0 \quad \forall\, \mathbf{v}_{ph} \in \mathbf{V}_{ph}. \tag{6.3}
$$

These operators have the approximation properties [27]:

$$
\|p_\star - Q_{\star h} p_\star\|_{L^2(\Omega_\star)} \le C h^{r_{s_\star}} \|p_\star\|_{H^{r_{s_\star}}(\Omega_\star)}, \qquad 0 \le r_{s_\star} \le s_\star + 1, \tag{6.4}
$$

$$
\|\lambda - Q_{\lambda h}\lambda\|_{L^2(\Gamma_{fp})} \le C h^{r_\lambda} \|\lambda\|_{H^{r_\lambda}(\Gamma_{fp})}, \qquad 0 \le r_\lambda \le k_p + 1. \tag{6.5}
$$

Next, we consider a Stokes–like projection operator [3], $(S_{fh}, R_{fh}) : \mathbf{V}_f \to \mathbf{V}_{fh} \times W_{fh}$, defined for all $\mathbf{v}_f \in \mathbf{V}_f$ by

$$
\begin{aligned}
&a_f(S_{fh}\mathbf{v}_f, \mathbf{v}_{fh}; c_h) - b_f(\mathbf{v}_{fh}, R_{fh}\mathbf{v}_f) = a_f(\mathbf{v}_f, \mathbf{v}_{fh}; c_h) && \forall\, \mathbf{v}_{fh} \in \mathbf{V}_{fh},\\
&b_f(S_{fh}\mathbf{v}_f, w_{fh}) = b_f(\mathbf{v}_f, w_{fh}) && \forall\, w_{fh} \in W_{fh}.
\end{aligned}
$$

Let $\Pi_{ph}$ be the mixed finite element interpolant onto $\mathbf{V}_{ph}$ satisfying for all $\mathbf{v}_p \in \mathbf{H}^1(\Omega_p)$ [8],

$$
\begin{aligned}
&(\nabla \cdot \Pi_{ph}\mathbf{v}_p, w_{ph}) = (\nabla \cdot \mathbf{v}_p, w_{ph}) && \forall w_{ph} \in W_{ph},\\
&\langle \Pi_{ph}\mathbf{v}_p \cdot \mathbf{n}_p, \mathbf{v}_{ph} \cdot \mathbf{n}_p \rangle_{\Gamma_{fp}} = \langle \mathbf{v}_p \cdot \mathbf{n}_p, \mathbf{v}_{ph} \cdot \mathbf{n}_p \rangle_{\Gamma_{fp}} && \forall \mathbf{v}_{ph} \in \mathbf{V}_{ph},\\
&\|\mathbf{v}_p - \Pi_{ph}\mathbf{v}_p\|_{\mathbf{L}^2(\Omega_p)} \le C h^{r_{k_p}} \|\mathbf{v}_p\|_{\mathbf{H}^{r_{k_p}}(\Omega_p)} && 1 \le r_{k_p} \le k_p + 1.
\end{aligned}
$$

Let $S_{sh}$ and $I_{ch}$ denote the Scott–Zhang interpolants from $\mathbf{X}_p$ onto $\mathbf{X}_{ph}$ and from $X_c$ onto $X_{ch}$, respectively, satisfying [50]:

$$\|\boldsymbol{\xi}_p - S_{sh}\boldsymbol{\xi}_p\|_{\mathbf{L}^2(\Omega_p)} + h\,|\boldsymbol{\xi}_p - S_{sh}\boldsymbol{\xi}_p|_{\mathbf{H}^1(\Omega_p)} \,\le\, C\,h^{r_{k_s}}\|\boldsymbol{\xi}_p\|_{\mathbf{H}^{r_{k_s}}(\Omega_p)} \quad 1 \le r_{k_s} \le k_s + 1\,. \tag{6.6}$$

$$\text{and} \quad \|\psi - I_{ch}\psi\|_{L^2(\Omega)} + h\,|\psi - I_{ch}\psi|_{H^1(\Omega)} \,\le\, C\,h^{r_{k_c}}\|\psi\|_{H^{r_{k_c}}(\Omega)} \quad 1 \le r_{k_c} \le k_c + 1\,. \tag{6.7}$$

Next, consider the space that satisfies the weak continuity of normal velocity condition:

$$\mathbf{U} = \Big\{(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p) \in \mathbf{V}_f \times \mathbf{H}^1(\Omega_p) \times \mathbf{X}_p : \quad b_\Gamma(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \mu) = 0 \quad \forall \mu \in \Lambda\Big\},$$

and its discrete version

$$\mathbf{U}_h = \Big\{(\mathbf{v}_{fh}, \mathbf{v}_{ph}, \boldsymbol{\xi}_{ph}) \in \mathbf{V}_{fh} \times \mathbf{V}_{ph} \times \mathbf{X}_{ph} : \quad b_\Gamma\left(\mathbf{v}_{fh}, \mathbf{v}_{ph}, \boldsymbol{\xi}_{ph}; \mu_h\right) = 0 \quad \forall \mu_h \in \tilde{\Lambda}_h\Big\}\,.$$

An interpolation operator $I_h : \mathbf{U} \to \mathbf{U}_h$ is constructed in [3, Section 5] as a triple

$$I_h(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p) = \left(I_{fh}\mathbf{v}_f, I_{ph}\mathbf{v}_p, I_{sh}\boldsymbol{\xi}_p\right),$$

where $I_{fh} = S_{fh}$, $I_{sh} = S_{sh}$, and $I_{ph}$ is based on a correction of $\Pi_{ph}$ designed to satisfy the continuity of normal velocity. The interpolant $I_h$ has the following properties:

$$b_\Gamma\left(I_{fh}\mathbf{v}_f, I_{ph}\mathbf{v}_p, I_{sh}\boldsymbol{\xi}_p; \mu_h\right) = 0 \quad \forall \mu_h \in \tilde{\Lambda}_h\,, \tag{6.8}$$

$$b_\star(I_{\star h}\mathbf{v}_\star - \mathbf{v}_\star, w_{\star h}) = 0 \quad \forall w_{\star h} \in W_{\star h}\,, \quad \star \in \{f, p\}\,. \tag{6.9}$$

The approximation properties of the components of $I_h$ are established in [3, Lemma 5.1]: for all sufficiently smooth $\mathbf{v}_f$, $\mathbf{v}_p$, and $\boldsymbol{\xi}_p$,

$$\|\mathbf{v}_f - I_{fh}\mathbf{v}_f\|_{\mathbf{H}^1(\Omega_f)} \le Ch^{r_{k_f}}\|\mathbf{v}_f\|_{\mathbf{H}^{r_{k_f}+1}(\Omega_f)}\,, \quad 0 \le r_{k_f} \le k_f\,. \tag{6.10}$$

$$\|\mathbf{v}_p - I_{ph}\mathbf{v}_p\|_{\mathbf{L}^2(\Omega_p)} \le C\Big(h^{r_{k_p}}\|\mathbf{v}_p\|_{\mathbf{H}^{r_{k_p}}(\Omega_p)} + h^{r_{k_f}}\|\mathbf{v}_f\|_{\mathbf{H}^{r_{k_f}+1}(\Omega_f)} + h^{r_{k_s}}\|\boldsymbol{\xi}_p\|_{\mathbf{H}^{r_{k_s}+1}(\Omega_p)}\Big)\,,$$
$$1 \le r_{k_p} \le k_p + 1\,,\ 0 \le r_{k_f} \le k_f\,,\ 0 \le r_{k_s} \le k_s\,. \tag{6.11}$$

$$\|\nabla\cdot(\mathbf{v}_p - I_{ph}\mathbf{v}_p)\|_{L^2(\Omega_p)} \le Ch^{r_{k_p}}\|\nabla\cdot\mathbf{v}_p\|_{\mathbf{H}^{r_{k_p}+1}(\Omega_p)}\,, \quad 0 \le r_{k_p} \le k_p\,. \tag{6.12}$$

$$\|\boldsymbol{\xi}_p - I_{sh}\boldsymbol{\xi}_p\|_{\mathbf{L}^2(\Omega_p)} + h|\boldsymbol{\xi}_p - I_{sh}\boldsymbol{\xi}_p|_{\mathbf{H}^1(\Omega_p)} \le Ch^{r_{k_s}}\|\boldsymbol{\xi}_p\|_{\mathbf{H}^{r_{k_s}}(\Omega_p)}\,, \quad 1 \le r_{k_s} \le k_s + 1\,. \tag{6.13}$$

### 6.2 Error estimates

We proceed with estimating the error between (**P**) and (**DP**). We introduce the errors for all variables and split them into approximation and discretization errors:

$$\mathbf{E}_{\mathbf{u}_f} := \mathbf{u}_f - \mathbf{u}_{fh} = (\mathbf{u}_f - I_{fh}\mathbf{u}_f) + (I_{fh}\mathbf{u}_f - \mathbf{u}_{fh}) := \boldsymbol{\chi}_{\mathbf{u}_f} + \boldsymbol{\phi}_{\mathbf{u}_f}\,, \tag{6.14a}$$

$$\mathbf{E}_{\mathbf{u}_p} := \mathbf{u}_p - \mathbf{u}_{ph} = (\mathbf{u}_p - I_{ph}\mathbf{u}_p) + (I_{ph}\mathbf{u}_p - \mathbf{u}_{ph}) := \boldsymbol{\chi}_{\mathbf{u}_p} + \boldsymbol{\phi}_{\mathbf{u}_p}\,, \tag{6.14b}$$

$$\mathbf{E}_{\boldsymbol{\eta}_p} := \boldsymbol{\eta}_p - \boldsymbol{\eta}_{ph} = (\boldsymbol{\eta}_p - I_{sh}\boldsymbol{\eta}_p) + (I_{sh}\boldsymbol{\eta}_p - \boldsymbol{\eta}_{ph}) := \boldsymbol{\chi}_{\boldsymbol{\eta}_p} + \boldsymbol{\phi}_{\boldsymbol{\eta}_p}\,, \tag{6.14c}$$

$$E_{p_f} := p_f - p_{fh} = (p_f - Q_{fh}p_f) + (Q_{fh}p_f - p_{fh}) := \chi_{p_f} + \phi_{p_f}\,, \tag{6.14d}$$

$$E_{p_p} := p_p - p_{ph} = (p_p - Q_{ph}p_p) + (Q_{ph}p_p - p_{ph}) := \chi_{p_p} + \phi_{p_p}\,, \tag{6.14e}$$

$$E_\lambda := \lambda - \lambda_h = (\lambda - Q_{\lambda h}\lambda) + (Q_{\lambda h}\lambda - \lambda_h) := \chi_\lambda + \phi_\lambda\,, \tag{6.14f}$$

$$E_c := c - c_h = (c - I_{ch}c) + (I_{ch}c - c_h) := \chi_c + \phi_c\,. \tag{6.14g}$$

In addition, we denote

$$\mathbf{E}_{\mathbf{u}} := \mathbf{u} - \mathbf{u}_h\,, \quad \text{with} \quad \mathbf{E}_{\mathbf{u}}|_{\Omega_*} = \mathbf{E}_{\mathbf{u}_*}\,, \quad * \in \{f, p\}\,. \tag{6.15}$$

**Theorem 6.1** *Under the conditions of Theorems 4.1 and 5.2, assuming that the solution of problem* ($\mathbf{P}$) *is sufficient smooth, then for the solution of the semi-discrete scheme* ($\mathbf{DP}$), *cf.* (5.3), *with initial data constructed in Theorem 3.2, there exist constants $C$ and $L$ independent of $h$, such that*

$$\begin{aligned}
&\|\boldsymbol{\eta}_p - \boldsymbol{\eta}_{ph}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_p))} + \sqrt{s_0}\|p_p - p_{ph}\|_{L^\infty(0,T;L^2(\Omega_p))} + \|\mathbf{u}_f - \mathbf{u}_{fh}\|_{L^2(0,T;\mathbf{H}^1(\Omega_f))} \\
&+ \|\lambda - \lambda_h\|_{L^2(0,T;L^2(\Gamma_{fp}))} + \|\mathbf{u}_p - \mathbf{u}_{ph}\|_{L^2(0,T;\mathbf{L}^2(\Omega_p))} + \left|(\mathbf{u}_f - \partial_t\boldsymbol{\eta}_p) - (\mathbf{u}_{fh} - \partial_t\boldsymbol{\eta}_{ph})\right|_{L^2(0,T;BJS)} \\
&+ \|p_f - p_{fh}\|_{L^2(0,T;L^2(\Omega_f))} + \|p_p - p_{ph}\|_{L^2(0,T;L^2(\Omega_p))} + \|c - c_h\|_{L^\infty(0,T;L^2(\Omega))} + \|c - c_h\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \\
&\leq C\sqrt{\exp(LT)}\, h^{\min\{k_f, s_f+1, k_p+1, s_p+1, k_s, k_c\}} . \qquad (6.16)
\end{aligned}$$

*Proof.* We first obtain an error estimate for the Stokes–Biot subproblem, which depends on $E_c$, followed by an error estimate for the transport subproblem, which depends on $\mathbf{E}_{\mathbf{u}}$, then combine the two to obtain the error estimate for the coupled problem.

**Error estimate for the Stokes–Biot subproblem.** Subtracting (5.3a)–(5.3f) from (2.11a)–(2.11f) and summing the equations, we obtain the following error equation:

$$\begin{aligned}
&a_f(\mathbf{E}_{\mathbf{u}_f}, \mathbf{v}_{fh}; c_h) + a_p^d(\mathbf{E}_{\mathbf{u}_p}, \mathbf{v}_{ph}; c_h) + a_p^e(\mathbf{E}_{\boldsymbol{\eta}_p}, \boldsymbol{\xi}_{ph}) + a_{BJS}(\mathbf{E}_{\mathbf{u}_f}, \partial_t\mathbf{E}_s; \mathbf{v}_{fh}, \boldsymbol{\xi}_{ph}) \\
&\quad + s_0(\partial_t E_{p_p}, w_{ph})_{\Omega_p} + b_f(\mathbf{v}_{fh}, E_{p_f}) + b_p(\mathbf{v}_{ph}, E_{p_p}) + \alpha b_e(\boldsymbol{\xi}_{ph}, E_{p_p}) \\
&\quad + b_\Gamma(\mathbf{v}_{fh}, \mathbf{v}_{ph}, \boldsymbol{\xi}_{ph}; E_\lambda) - b_f(\mathbf{E}_{\mathbf{u}_f}, w_{fh}) - b_p(\mathbf{E}_{\mathbf{u}_p}, w_{ph}) - \alpha b_e(\partial_t\mathbf{E}_{\boldsymbol{\eta}_p}, w_{ph}) \\
&= -(2(\mu(c) - \mu(c_h))\boldsymbol{\epsilon}(\mathbf{u}_f), \boldsymbol{\epsilon}(\mathbf{v}_{fh}))_{\Omega_f} - ((\mu(c) - \mu(c_h))K^{-1}\mathbf{u}_p, \mathbf{v}_{ph})_{\Omega_p} . \qquad (6.17)
\end{aligned}$$

We set $\mathbf{v}_{fh} = \boldsymbol{\phi}_{\mathbf{u}_f}$, $w_{fh} = \phi_{p_f}$, $\boldsymbol{\xi}_{ph} = \partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p}$, $\mathbf{v}_{ph} = \boldsymbol{\phi}_{\mathbf{u}_p}$, $w_{ph} = \phi_{p_p}$ in (6.17) and split the error as the approximation and discretization error as shown in (6.14). Some terms can be simplified due to the projection properties (6.1), (6.8), and (6.9):

$$b_f(\boldsymbol{\chi}_{\mathbf{u}_f}, \phi_{p_f}) = b_p(\boldsymbol{\chi}_{\mathbf{u}_p}, \phi_{p_p}) = b_p(\boldsymbol{\phi}_{\mathbf{u}_p}, \chi_{p_p}) = (\chi_{p_p}, \phi_{p_p})_{\Omega_p} = b_\Gamma(\boldsymbol{\phi}_{\mathbf{u}_f}, \boldsymbol{\phi}_{\mathbf{u}_p}, \partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p}; \phi_\lambda) = 0 .$$

We also use (6.3) to obtain

$$\left\langle \boldsymbol{\phi}_{\mathbf{u}_p} \cdot \mathbf{n}_p, \chi_\lambda \right\rangle_{\Gamma_{fp}} = 0 . \qquad (6.18)$$

Using the above simplifications, (6.17) results in

$$\begin{aligned}
&a_f(\boldsymbol{\phi}_{\mathbf{u}_f}, \boldsymbol{\phi}_{\mathbf{u}_f}; c_h) + a_p^d(\boldsymbol{\phi}_{\mathbf{u}_p}, \boldsymbol{\phi}_{\mathbf{u}_p}; c_h) + |\boldsymbol{\phi}_{\mathbf{u}_f} - \partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p}|^2_{BJS} + \frac{1}{2}\frac{d}{dt}a_p^e(\boldsymbol{\phi}_{\boldsymbol{\eta}_p}, \boldsymbol{\phi}_{\boldsymbol{\eta}_p}) + \frac{s_0}{2}\frac{d}{dt}\|\phi_{p_p}\|^2_{L^2(\Omega)} \\
&= -a_f(\boldsymbol{\chi}_{\mathbf{u}_f}, \boldsymbol{\phi}_{\mathbf{u}_f}; c_h) - a_p^e(\boldsymbol{\chi}_{\boldsymbol{\eta}_p}, \partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p}) - a_p^d(\boldsymbol{\chi}_{\mathbf{u}_p}, \boldsymbol{\phi}_{\mathbf{u}_p}; c_h) - a_{BJS}(\boldsymbol{\chi}_{\mathbf{u}_f}, \partial_t\boldsymbol{\chi}_{\boldsymbol{\eta}_p}; \boldsymbol{\phi}_{\mathbf{u}_f}, \partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p}) \\
&\quad - b_f(\boldsymbol{\phi}_{\mathbf{u}_f}, \chi_{p_f}) - \alpha b_e(\partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p}, \chi_{p_p}) + \alpha b_e(\partial_t\boldsymbol{\chi}_{\boldsymbol{\eta}_p}, \phi_{p_p}) - \langle \boldsymbol{\phi}_{\mathbf{u}_f} \cdot \mathbf{n}_f + \partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p} \cdot \mathbf{n}_p, \chi_\lambda\rangle_{\Gamma_{fp}} \\
&\quad - (2(\mu(c) - \mu(c_h))\boldsymbol{\epsilon}(\mathbf{u}_f), \boldsymbol{\epsilon}(\boldsymbol{\phi}_{\mathbf{u}_f}))_{\Omega_f} - ((\mu(c) - \mu(c_h))K^{-1}\mathbf{u}_p, \boldsymbol{\phi}_{\mathbf{u}_p})_{\Omega_p} . \qquad (6.19)
\end{aligned}$$

We proceed with bounding the terms on the right hand side of (6.19). Using (5.10) and the Cauchy–Schwarz, trace, and Young's inequalities, we obtain, for any $\epsilon > 0$,

$$\begin{aligned}
&\left|a_f(\boldsymbol{\chi}_{\mathbf{u}_f}, \boldsymbol{\phi}_{\mathbf{u}_f}; c_h)\right| + \left|a_p^d(\boldsymbol{\chi}_{\mathbf{u}_p}, \boldsymbol{\phi}_{\mathbf{u}_p}; c_h)\right| + \left|a_{BJS}(\boldsymbol{\chi}_{\mathbf{u}_f}, \partial_t\boldsymbol{\chi}_{\boldsymbol{\eta}_p}; \boldsymbol{\phi}_{\mathbf{u}_f}, \partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p})\right| \\
&\leq \epsilon\left(\|\boldsymbol{\phi}_{\mathbf{u}_f}\|^2_{\mathbf{H}^1(\Omega_f)} + \|\boldsymbol{\phi}_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} + |\boldsymbol{\phi}_{\mathbf{u}_f} - \partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p}|^2_{BJS}\right) \\
&\quad + \frac{C}{\epsilon}\left(\|\boldsymbol{\chi}_{\mathbf{u}_f}\|^2_{\mathbf{H}^1(\Omega_f)} + \|\boldsymbol{\chi}_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} + \|\partial_t\boldsymbol{\chi}_{\boldsymbol{\eta}_p}\|^2_{\mathbf{H}^1(\Omega_p)}\right) . \qquad (6.20)
\end{aligned}$$

For the viscous terms on the right hand side of (6.19), using (4.35), we deduce, for some $\epsilon > 0$,

$$\begin{aligned}
&\left|\big(2(\mu(c)-\mu(c_h))\boldsymbol{\epsilon}(\mathbf{u}_f),\boldsymbol{\epsilon}(\boldsymbol{\phi}_{\mathbf{u}_f})\big)_{\Omega_f}\right| + \left|\big((\mu(c)-\mu(c_h))K^{-1}\mathbf{u}_p,\boldsymbol{\phi}_{\mathbf{u}_p}\big)_{\Omega_p}\right| \\
&\qquad \le \epsilon\left(\|\boldsymbol{\phi}_{\mathbf{u}_f}\|^2_{\mathbf{H}^1(\Omega_f)} + \|\boldsymbol{\phi}_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)}\right) + \frac{C}{\epsilon}\|E_c\|^2_{L^2(\Omega)}\,. \qquad (6.21)
\end{aligned}$$

We bound the rest terms on the right hand side of (6.19) that do not involve $\partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p}$ as follows:

$$\begin{aligned}
&\left|b_f(\boldsymbol{\phi}_{\mathbf{u}_f},\chi_{p_f})\right| + \left|\alpha b_e(\partial_t\boldsymbol{\chi}_{\boldsymbol{\eta}_p},\phi_{p_p})\right| + \left|\langle\boldsymbol{\phi}_{\mathbf{u}_f}\cdot\mathbf{n}_f,\chi_\lambda\rangle_{\Gamma_{fp}}\right| \\
&\le \epsilon\left(\|\boldsymbol{\phi}_{\mathbf{u}_f}\|^2_{\mathbf{H}^1(\Omega_f)} + \|\phi_{p_p}\|^2_{L^2(\Omega_p)}\right) + \frac{C}{\epsilon}\left(\|\chi_{p_f}\|^2_{L^2(\Omega_f)} + \|\partial_t\boldsymbol{\chi}_{\boldsymbol{\eta}_p}\|^2_{\mathbf{H}^1(\Omega_p)} + \|\chi_\lambda\|^2_{L^2(\Gamma_{fp})}\right), \quad (6.22)
\end{aligned}$$

where we used the trace inequality for the third term. Next, after time integration, we use integration by parts in time for the terms on the right hand side of (6.19) containing $\partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p}$:

$$\begin{aligned}
&-\int_0^t\left(a_p^e(\boldsymbol{\chi}_{\boldsymbol{\eta}_p},\partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p}) + \alpha b_e(\partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p},\chi_{p_p}) + \langle\partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p}\cdot\mathbf{n}_p,\chi_\lambda\rangle_{\Gamma_{fp}}\right)ds \\
&= -a_p^e(\boldsymbol{\chi}_{\boldsymbol{\eta}_p},\boldsymbol{\phi}_{\boldsymbol{\eta}_p})\Big|_0^t - \alpha b_e(\boldsymbol{\phi}_{\boldsymbol{\eta}_p},\chi_{p_p})\Big|_0^t - \langle\boldsymbol{\phi}_{\boldsymbol{\eta}_p}\cdot\mathbf{n}_p,\chi_\lambda\rangle_{\Gamma_{fp}}\Big|_0^t \\
&\quad + \int_0^t\left(a_p^e(\partial_t\boldsymbol{\chi}_{\boldsymbol{\eta}_p},\boldsymbol{\phi}_{\boldsymbol{\eta}_p}) + \alpha b_e(\boldsymbol{\phi}_{\boldsymbol{\eta}_p},\partial_t\chi_{p_p}) + \langle\boldsymbol{\phi}_{\boldsymbol{\eta}_p}\cdot\mathbf{n}_p,\partial_t\chi_\lambda\rangle_{\Gamma_{fp}}\right)ds \\
&\le \epsilon\|\boldsymbol{\phi}_{\boldsymbol{\eta}_p}(t)\|^2_{\mathbf{H}^1(\Omega_p)} + \frac{C}{\epsilon}\left(\|\boldsymbol{\chi}_{\boldsymbol{\eta}_p}(t)\|^2_{\mathbf{H}^1(\Omega_p)} + \|\chi_{p_p}(t)\|^2_{L^2(\Omega_p)} + \|\chi_\lambda(t)\|^2_{L^2(\Gamma_{fp})}\right) \\
&\quad + C\left(\|\boldsymbol{\phi}_{\boldsymbol{\eta}_p}(0)\|^2_{\mathbf{H}^1(\Omega_p)} + \|\boldsymbol{\chi}_{\boldsymbol{\eta}_p}(0)\|^2_{\mathbf{H}^1(\Omega_p)} + \|\chi_{p_p}(0)\|^2_{L^2(\Omega_p)} + \|\chi_\lambda(0)\|^2_{L^2(\Gamma_{fp})}\right) \\
&\quad + C\int_0^t\left(\|\partial_t\boldsymbol{\chi}_{\boldsymbol{\eta}_p}\|^2_{\mathbf{H}^1(\Omega_p)} + \|\partial_t\chi_{p_p}\|^2_{L^2(\Omega_p)} + \|\partial_t\chi_\lambda\|^2_{L^2(\Gamma_{fp})} + \|\boldsymbol{\phi}_{\boldsymbol{\eta}_p}\|^2_{\mathbf{H}^1(\Omega_p)}\right)ds\,. \qquad (6.23)
\end{aligned}$$

Integrating (6.19) in time from 0 to $t$, combining it with (6.20)–(6.23), utilizing the coercivity of the functionals on the left hand side of (6.19), cf. Lemma 2.1, and taking $\epsilon$ small enough, we obtain

$$\begin{aligned}
&\|\boldsymbol{\phi}_{\boldsymbol{\eta}_p}(t)\|^2_{\mathbf{H}^1(\Omega_p)} + s_0\|\phi_{p_p}(t)\|^2_{L^2(\Omega_p)} + \int_0^t\left(\|\boldsymbol{\phi}_{\mathbf{u}_f}\|^2_{\mathbf{H}^1(\Omega_f)} + \|\boldsymbol{\phi}_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} + |\boldsymbol{\phi}_{\mathbf{u}_f} - \partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p}|^2_{BJS}\right)ds \\
&\le C\Bigg(\int_0^t\Big(\|\boldsymbol{\chi}_{\mathbf{u}_f}\|^2_{\mathbf{H}^1(\Omega_f)} + \|\boldsymbol{\chi}_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} + \|\partial_t\boldsymbol{\chi}_{\boldsymbol{\eta}_p}\|^2_{\mathbf{H}^1(\Omega_p)} + \|\chi_{p_f}\|^2_{L^2(\Omega_f)} + \|\chi_\lambda\|^2_{L^2(\Gamma_{fp})} \\
&\quad + \|\partial_t\chi_{p_p}\|^2_{L^2(\Omega_p)} + \|\partial_t\chi_\lambda\|^2_{L^2(\Gamma_{fp})}\Big)ds + \|\boldsymbol{\chi}_{\boldsymbol{\eta}_p}(t)\|^2_{\mathbf{H}^1(\Omega_p)} + \|\chi_{p_p}(t)\|^2_{L^2(\Omega_p)} + \|\chi_\lambda(t)\|^2_{L^2(\Gamma_{fp})} \\
&\quad + \|\boldsymbol{\chi}_{\boldsymbol{\eta}_p}(0)\|^2_{\mathbf{H}^1(\Omega_p)} + \|\chi_{p_p}(0)\|^2_{L^2(\Omega_p)} + \|\chi_\lambda(0)\|^2_{L^2(\Gamma_{fp})} + \|\boldsymbol{\phi}_{\boldsymbol{\eta}_p}(0)\|^2_{\mathbf{H}^1(\Omega_p)} + \|\phi_{p_p}(0)\|^2_{L^2(\Omega_p)} \\
&\quad + \int_0^t\left(\|\boldsymbol{\phi}_{\boldsymbol{\eta}_p}\|_{\mathbf{H}^1(\Omega_p)} + \|E_c\|^2_{L^2(\Omega)}\right)ds\Bigg) + \epsilon\int_0^t\|\phi_{p_p}\|^2_{L^2(\Omega_p)}\,ds\,. \qquad (6.24)
\end{aligned}$$

Next, we use the discrete inf-sup condition (5.2) to bound $\phi_{p_f}$, $\phi_{p_p}$ and $\phi_\lambda$. From (6.17), we have

$$\begin{aligned}
&\|(\phi_{p_f},\phi_{p_p},\phi_\lambda)\|_{W_f\times W_p\times\tilde{\Lambda}_h} \\
&\quad \le C\sup_{0\neq(\mathbf{v}_{fh},\mathbf{v}_{ph})\in\mathbf{V}_f\times\mathbf{V}_p}\|(\mathbf{v}_{fh},\mathbf{v}_{ph})\|^{-1}_{\mathbf{V}_f\times\mathbf{V}_p}\Big(-a_f(\mathbf{E}_{\mathbf{u}_f},\mathbf{v}_{fh};c_h) - a_p^d(\mathbf{E}_{\mathbf{u}_p},\mathbf{v}_{ph};c_h)
\end{aligned}$$

$$
\begin{aligned}
&\quad - a_{BJS}(\mathbf{E}_{\mathbf{u}_f}, \partial_t \mathbf{E}_{\boldsymbol{\eta}_p}; \mathbf{v}_{fh}, \mathbf{0}) - b_f(\mathbf{v}_{fh}, \chi_{p_f}) - b_p(\mathbf{v}_{ph}, \chi_{p_p}) - b_\Gamma(\mathbf{v}_{fh}, \mathbf{v}_{ph}, \mathbf{0}; \chi_\lambda) \\
&\quad - (2(\mu(c) - \mu(c_h))\boldsymbol{\epsilon}(\mathbf{u}_f), \boldsymbol{\epsilon}(\mathbf{v}_{fh}))_{\Omega_f} - ((\mu(c) - \mu(c_h))K^{-1}\mathbf{u}_p, \mathbf{v}_{ph})_{\Omega_p}\Big) \\
&\leq C\Big(\|\boldsymbol{\phi}_{\mathbf{u}_f}\|_{\mathbf{H}^1(\Omega_f)} + \|\boldsymbol{\phi}_{\mathbf{u}_p}\|_{\mathbf{L}^2(\Omega_p)} + |\boldsymbol{\phi}_{\mathbf{u}_f} - \partial_t \boldsymbol{\phi}_{\boldsymbol{\eta}_p}|_{BJS} + \|\boldsymbol{\chi}_{\mathbf{u}_f}\|_{\mathbf{H}^1(\Omega_f)} + \|\boldsymbol{\chi}_{\mathbf{u}_p}\|_{\mathbf{L}^2(\Omega_p)} \\
&\quad + \|\partial_t \boldsymbol{\chi}_{\boldsymbol{\eta}_p}\|_{\mathbf{H}^1(\Omega_p)} + \|\chi_{p_f}\|_{L^2(\Omega_f)} + \|\chi_{p_p}\|_{L^2(\Omega_p)} + \|\chi_\lambda\|_{L^2(\Gamma_{pf})} + \|E_c\|_{L^2(\Omega)}\Big), \qquad (6.25)
\end{aligned}
$$

where we used (6.3) in the last inequality. Combining (6.24) with the squared and time-integrated (6.25) and taking $\epsilon$ small enough, results in

$$
\begin{aligned}
&\|\boldsymbol{\phi}_{\boldsymbol{\eta}_p}(t)\|^2_{\mathbf{H}^1(\Omega_p)} + s_0\|\phi_{p_p}(t)\|^2_{L^2(\Omega_p)} + \int_0^t \Big(\|\boldsymbol{\phi}_{\mathbf{u}_f}\|^2_{\mathbf{H}^1(\Omega_f)} + \|\boldsymbol{\phi}_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} + |\boldsymbol{\phi}_{\mathbf{u}_f} - \partial_t \boldsymbol{\phi}_{\boldsymbol{\eta}_p}|^2_{BJS}\Big)\, ds \\
&\quad + \int_0^t \Big(\|\phi_{p_f}\|^2_{L^2(\Omega_f)} + \|\phi_{p_p}\|^2_{L^2(\Omega_p)} + \|\phi_\lambda\|^2_{L^2(\Gamma_{fp})}\Big)\, ds \\
&\leq C\Bigg(\int_0^t \Big(\|\boldsymbol{\chi}_{\mathbf{u}_f}\|^2_{\mathbf{H}^1(\Omega_f)} + \|\boldsymbol{\chi}_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} + \|\partial_t \boldsymbol{\chi}_{\boldsymbol{\eta}_p}\|^2_{\mathbf{H}^1(\Omega_p)} + \|\chi_{p_f}\|^2_{L^2(\Omega_f)} \qquad (6.26) \\
&\quad + \|\chi_{p_p}\|^2_{L^2(\Omega_p)} + \|\chi_\lambda\|^2_{L^2(\Gamma_{fp})} + \|\partial_t \chi_{p_p}\|^2_{L^2(\Omega_p)} + \|\partial_t \chi_\lambda\|^2_{L^2(\Gamma_{fp})}\Big) ds + \|\boldsymbol{\chi}_{\boldsymbol{\eta}_p}(t)\|^2_{\mathbf{H}^1(\Omega_p)} \\
&\quad + \|\chi_{p_p}(t)\|^2_{L^2(\Omega_p)} + \|\chi_\lambda(t)\|^2_{L^2(\Gamma_{fp})} + \|\boldsymbol{\chi}_{\boldsymbol{\eta}_p}(0)\|^2_{\mathbf{H}^1(\Omega_p)} + \|\chi_{p_p}(0)\|^2_{L^2(\Omega_p)} + \|\chi_\lambda(0)\|^2_{L^2(\Gamma_{fp})} \\
&\quad + \|\boldsymbol{\phi}_{\boldsymbol{\eta}_p}(0)\|^2_{\mathbf{H}^1(\Omega_p)} + \|\phi_{p_p}(0)\|^2_{L^2(\Omega_p)} + \int_0^t \Big(\|\boldsymbol{\phi}_{\boldsymbol{\eta}_p}\|^2_{\mathbf{H}^1(\Omega_p)} + \|E_c\|^2_{L^2(\Omega)}\Big)\, ds\Bigg).
\end{aligned}
$$

**Error estimate for the transport subproblem.** We next derive an estimate for $E_c$, which depends on $\mathbf{E}_{\mathbf{u}}$. We take the difference between (2.11g) and (5.3g) to obtain the error equation

$$
\begin{aligned}
&(\phi\partial_t E_c, \psi_h) + (\mathbf{D}(\mathbf{u}_h)\nabla E_c, \nabla\psi_h) + ((\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}_h)\nabla c, \nabla\psi_h) \\
&\quad - (c\mathbf{E}_{\mathbf{u}}, \nabla\psi_h) - (E_c\mathbf{u}_h, \nabla\psi_h) - (q^- E_c, \psi_h) = 0\,. \qquad (6.27)
\end{aligned}
$$

We take the test function $\psi_h = \phi_c$, and rearrange (6.27) as follows:

$$
\begin{aligned}
&(\phi\partial_t\phi_c, \phi_c) + (\mathbf{D}(\mathbf{u}_h)\nabla\phi_c, \nabla\phi_c) - (q^-\phi_c, \phi_c) \\
&\quad = -(\phi\partial_t\chi_c, \phi_c) - (\mathbf{D}(\mathbf{u}_h)\nabla\chi_c, \nabla\phi_c) + ((\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}_h))\nabla c, \nabla\phi_c) \\
&\qquad + (\phi_c\mathbf{u}_h, \nabla\phi_c) + (\chi_c\mathbf{u}_h, \nabla\phi_c) + (c\mathbf{E}_{\mathbf{u}}, \nabla\phi_c) + (q^-\chi_c, \phi_c)\,.
\end{aligned}
$$

Using (4.35), (5.10), (2.15), and the Cauchy–Schwarz and Young's inequalities, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi^{1/2}\phi_c\|^2_{L^2(\Omega)} + \|\mathbf{D}(\mathbf{u}_h)^{1/2}\nabla\phi_c\|^2_{\mathbf{L}^2(\Omega)} \\
&\quad \leq \epsilon\|\nabla\phi_c\|^2_{\mathbf{L}^2(\Omega)} + \frac{C}{\epsilon}\Big(\|\phi_c\|^2_{L^2(\Omega)} + \|\chi_c\|^2_{L^2(\Omega)} + \|\partial_t\chi_c\|^2_{L^2(\Omega)} + \|\nabla\chi_c\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{E}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)}\Big). \qquad (6.28)
\end{aligned}
$$

Integrating (6.28) from 0 to $t$ and taking $\epsilon$ small enough, we obtain

$$
\begin{aligned}
&\|\phi_c(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla\phi_c\|^2_{\mathbf{L}^2(\Omega)}\, ds \,\leq\, C\bigg(\int_0^t \Big(\|\phi_c\|^2_{L^2(\Omega)} + \|\chi_c\|^2_{L^2(\Omega)} + \|\partial_t\chi_c\|^2_{L^2(\Omega)} \\
&\quad + \|\nabla\chi_c\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{E}_{\mathbf{u}_f}\|^2_{\mathbf{L}^2(\Omega_f)} + \|\mathbf{E}_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)}\Big) ds + \|\phi_c(0)\|^2_{L^2(\Omega)}\bigg). \qquad (6.29)
\end{aligned}
$$

**Error estimate for the coupled problem.** Combining (6.26) with a sufficiently small multiple of (6.29) and using Gronwall's inequality for the terms $\int_0^t \left( \|\boldsymbol{\phi}_{\boldsymbol{\eta}_p}\|^2_{\mathbf{H}^1(\Omega_p)} + \|E_c\|^2_{L^2(\Omega)} \right) ds$ in (6.26) and $\int_0^t \|\phi_c\|^2_{L^2(\Omega)} ds$ in (6.29), we obtain

$$
\begin{aligned}
&\|\boldsymbol{\phi}_{\boldsymbol{\eta}_p}(t)\|^2_{\mathbf{H}^1(\Omega_p)} + s_0\|\phi_{p_p}(t)\|^2_{L^2(\Omega_p)} + \|\phi_c(t)\|^2_{L^2(\Omega)} + \int_0^t \Big( \|\boldsymbol{\phi}_{\mathbf{u}_f}\|^2_{\mathbf{H}^1(\Omega_f)} + \|\boldsymbol{\phi}_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} \\
&\quad + |\boldsymbol{\phi}_{\mathbf{u}_f} - \partial_t \boldsymbol{\phi}_{\boldsymbol{\eta}_p}|^2_{BJS} + \|\phi_{p_f}\|^2_{L^2(\Omega_f)} + \|\phi_{p_p}\|^2_{L^2(\Omega_p)} + \|\phi_\lambda\|^2_{L^2(\Gamma_{fp})} + \|\nabla\phi_c\|^2_{\mathbf{L}^2(\Omega)} \Big) ds \\
&\le C\,\exp(Lt) \bigg( \int_0^t \Big( \|\boldsymbol{\chi}_{\mathbf{u}_f}\|^2_{\mathbf{H}^1(\Omega_f)} + \|\boldsymbol{\chi}_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} + \|\partial_t\boldsymbol{\chi}_{\boldsymbol{\eta}_p}\|^2_{\mathbf{H}^1(\Omega_p)} + \|\chi_{p_f}\|^2_{L^2(\Omega_f)} + \|\chi_{p_p}\|^2_{L^2(\Omega_p)} \\
&\quad + \|\chi_\lambda\|^2_{L^2(\Gamma_{fp})} + \|\partial_t\chi_{p_p}\|^2_{L^2(\Omega_p)} + \|\partial_t\chi_\lambda\|^2_{L^2(\Gamma_{fp})} + \|\chi_c\|^2_{L^2(\Omega)} + \|\partial_t\chi_c\|^2_{L^2(\Omega)} + \|\nabla\chi_c\|^2_{\mathbf{L}^2(\Omega)} \Big) ds \\
&\quad + \|\boldsymbol{\chi}_{\boldsymbol{\eta}_p}(t)\|^2_{\mathbf{H}^1(\Omega_p)} + \|\chi_{p_p}(t)\|^2_{L^2(\Omega_p)} + \|\chi_\lambda(t)\|^2_{L^2(\Gamma_{fp})} + \|\boldsymbol{\chi}_{\boldsymbol{\eta}_p}(0)\|^2_{\mathbf{H}^1(\Omega_p)} + \|\chi_{p_p}(0)\|^2_{L^2(\Omega_p)} \\
&\quad + \|\chi_\lambda(0)\|^2_{L^2(\Gamma_{fp})} + \|\boldsymbol{\phi}_{\boldsymbol{\eta}_p}(0)\|^2_{\mathbf{H}^1(\Omega_p)} + \|\phi_{p_p}(0)\|^2_{L^2(\Omega_p)} + \|\phi_c(0)\|^2_{L^2(\Omega)} \bigg). \qquad (6.30)
\end{aligned}
$$

We next bound the initial error terms in (6.30). Recall that the Stokes-Biot initial data is constructed in (3.18) and (3.19) and the concentration initial data $c_{h,0}$ is defined in (3.9). Classical error analysis for the Stokes–Darcy system (3.18), see [42], and the elasticity problem (3.19), combined with the approximation properties (6.4)–(6.5) and (6.10)–(6.13), implies that

$$\|\boldsymbol{\phi}_{\boldsymbol{\eta}_p}(0)\|_{\mathbf{H}^1(\Omega_p)} + \|\phi_{p_p}(0)\|_{L^2(\Omega_p)} \le C\, h^{\min\{k_f, s_f+1, k_p+1, s_p+1, k_s\}}, \qquad (6.31)$$

while (3.9) gives that

$$\|\phi_c(0)\|_{L^2(\Omega)} = 0. \qquad (6.32)$$

The final error bound (6.16) follows from combining (6.30)–(6.32) and using the approximation properties (6.4)–(6.5), (6.7), and (6.10)–(6.13). □

**Remark 6.1** *The choice of the non-conforming Lagrange multiplier space $\tilde{\Lambda}_h$ (5.1) guarantees that (6.18) holds. If instead a conforming space $\Lambda_h \subset \Lambda$ was used, (6.18) would not hold. An optimal bound for the extra term $\langle \boldsymbol{\phi}_{\mathbf{u}_p} \cdot \mathbf{n}_p, \chi_\lambda \rangle_{\Gamma_{fp}}$, using the normal trace inequality, requires a bound on $\|\nabla\cdot\boldsymbol{\phi}_{\mathbf{u}_p}\|_{L^2(\Omega_p)}$, which is not provided in the above theorem. This would result in a reduction of $h^{-1/2}$ in the error estimate.*

We next establish a bound on $\|\nabla\cdot(\mathbf{u}_p - \mathbf{u}_{ph})\|_{L^2(\Omega_p)}$ under the following additional assumption.

**Assumption 6.1** *There exists a constant $\tilde{c} > 0$ such that $|\mathbf{u}| \ge \tilde{c}$ for all $(\mathbf{x}, t) \in \Omega_p \times (0, T]$.*

**Lemma 6.2** *Under Assumption 6.1, there exists a constant $C$ independent of $h$ such that for $1 \le i, j, k \le d$,*

$$\forall\, (\mathbf{x}, t) \in \Omega_p \times (0, T], \quad \left| \frac{\partial(\mathbf{D}(\mathbf{u}))_{i,j}}{\partial \mathbf{u}_k} - \frac{\partial(\mathbf{D}(\mathbf{u}_h))_{i,j}}{\partial \mathbf{u}_{h,k}} \right| \le C|\mathbf{u} - \mathbf{u}_h|. \qquad (6.33)$$

*Proof.* Consider the case $k = i \ne j$. The other cases are handled similarly. Recalling the definition (2.9), it is enough to consider $(\tilde{\mathbf{D}}(\mathbf{u}))_{i,j} := |\mathbf{u}|(\mathbf{E}(\mathbf{u}))_{i,j} = \frac{\mathbf{u}_i\mathbf{u}_j}{|\mathbf{u}|}$. We have

$$\frac{\partial(\tilde{\mathbf{D}}(\mathbf{u}))_{i,j}}{\partial \mathbf{u}_i} = \frac{\mathbf{u}_j}{|\mathbf{u}|} - \frac{\mathbf{u}_i^2\mathbf{u}_j}{|\mathbf{u}|^3}. \qquad (6.34)$$

For the first term on the right hand side we have

$$\left|\frac{\mathbf{u}_j}{|\mathbf{u}|}-\frac{\mathbf{u}_{h,j}}{|\mathbf{u}_h|}\right| = \left|\frac{\mathbf{u}_j-\mathbf{u}_{h,j}}{|\mathbf{u}|}+\frac{\mathbf{u}_{h,j}(|\mathbf{u}_h|-|\mathbf{u}|)}{|\mathbf{u}||\mathbf{u}_h|}\right| \le \left|\frac{\mathbf{u}_j-\mathbf{u}_{h,j}}{|\mathbf{u}|}\right| + \left|\frac{|\mathbf{u}_h|-|\mathbf{u}|}{|\mathbf{u}|}\right| \le \frac{2}{\tilde c}|\mathbf{u}-\mathbf{u}_h|\,, \tag{6.35}$$

where we used Assumption 6.1 and the reverse triangle inequality. For the second term on the right hand side of (6.34), assume without loss of generality that for a fixed $(\mathbf{x},t)\in\Omega_p\times(0,T]$, $|\mathbf{u}_h|\le|\mathbf{u}|$. If this is not true, then $|\mathbf{u}_h|\ge\tilde c$ and the argument can be done by reversing $\mathbf{u}$ and $\mathbf{u}_h$. We have

$$\left|\frac{\mathbf{u}_i^2\mathbf{u}_j}{|\mathbf{u}|^3}-\frac{\mathbf{u}_{h,i}^2\mathbf{u}_{h,j}}{|\mathbf{u}_h|^3}\right| = \left|\frac{\mathbf{u}_i^2(\mathbf{u}_j-\mathbf{u}_{h,j})}{|\mathbf{u}|^3}+\frac{(\mathbf{u}_i^2-\mathbf{u}_{h,i}^2)\mathbf{u}_{h,j}}{|\mathbf{u}|^3}+\frac{\mathbf{u}_{h,i}^2\mathbf{u}_{h,j}(|\mathbf{u}_h|^3-|\mathbf{u}|^3)}{|\mathbf{u}|^3|\mathbf{u}_h|^3}\right| \le \frac{6}{\tilde c}|\mathbf{u}-\mathbf{u}_h|\,. \tag{6.36}$$

Inequality (6.33) follows from (6.34)–(6.36). □

**Theorem 6.3** *Under the conditions of Theorem 6.1 and Assumption 6.1, assuming that the solution of problem* $(\mathbf{P})$ *is sufficiently smooth, there exist constants $C$ and $L$ independent of $h$, such that*

$$\|\nabla\cdot(\mathbf{u}_p-\mathbf{u}_{ph})\|_{L^\infty(0,T;L^2(\Omega_p))} \;\le\; C\,\sqrt{\exp(LT)}\,h^{\min\{k_f,s_f+1,k_p+1,s_p+1,k_s,k_c\}}\,. \tag{6.37}$$

*Proof.* We subtract (5.3e) from (2.11e), take $w_{ph}=\nabla\cdot\boldsymbol{\phi}_{\mathbf{u}_p}$, and use (6.9) to obtain

$$\|\nabla\cdot\boldsymbol{\phi}_{\mathbf{u}_p}\|_{L^2(\Omega_p)} \le s_0\|\partial_t\phi_{p_p}\|_{L^2(\Omega_p)} + s_0\|\partial_t\chi_{p_p}\|_{L^2(\Omega_p)} + \alpha\|\partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p}\|_{\mathbf{H}^1(\Omega_p)} + \alpha\|\partial_t\boldsymbol{\chi}_{\boldsymbol{\eta}_p}\|_{\mathbf{H}^1(\Omega_p)}\,. \tag{6.38}$$

We proceed by deriving bounds on $\|\partial_t\phi_{p_p}\|_{L^2(\Omega_p)}$ and $\|\partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p}\|_{\mathbf{H}^1(\Omega_p)}$. Differentiating in time the error equation (6.17) and choosing the test functions to be $\mathbf{v}_{fh}=\partial_t\boldsymbol{\phi}_{\mathbf{u}_f}$, $w_{fh}=\partial_t\phi_{p_f}$, $\boldsymbol{\eta}_p=\partial_{tt}\boldsymbol{\phi}_{\boldsymbol{\eta}_p}$, $\mathbf{v}_{ph}=\partial_t\boldsymbol{\phi}_{\mathbf{u}_p}$, $w_{ph}=\partial_t\phi_{p_p}$, and $\nu_h=\partial_t\phi_\lambda$, we obtain

$$\begin{aligned}
& a_f(\partial_t\boldsymbol{\phi}_{\mathbf{u}_f},\partial_t\boldsymbol{\phi}_{\mathbf{u}_f};c_h) + a_p^d(\partial_t\boldsymbol{\phi}_{\mathbf{u}_p},\partial_t\boldsymbol{\phi}_{\mathbf{u}_p};c_h) + |\partial_t\boldsymbol{\phi}_{\mathbf{u}_f}-\partial_{tt}\boldsymbol{\phi}_{\boldsymbol{\eta}_p}|^2_{BJS} \\
&\quad + a_p^e(\partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p},\partial_{tt}\boldsymbol{\phi}_{\boldsymbol{\eta}_p}) + s_0(\partial_{tt}\phi_{p_p},\partial_t\phi_{p_p})_{\Omega_p} \\
&= -a_f(\partial_t\boldsymbol{\chi}_{\mathbf{u}_f},\partial_t\boldsymbol{\phi}_{\mathbf{u}_f};c_h) - a_f(\boldsymbol{\phi}_{\mathbf{u}_f},\partial_t\boldsymbol{\phi}_{\mathbf{u}_f};\partial_t c_h) - a_f(\boldsymbol{\chi}_{\mathbf{u}_f},\partial_t\boldsymbol{\phi}_{\mathbf{u}_f};\partial_t c_h) - a_p^e(\partial_t\boldsymbol{\chi}_{\boldsymbol{\eta}_p},\partial_{tt}\boldsymbol{\phi}_{\boldsymbol{\eta}_p}) \\
&\quad - a_p^d(\partial_t\boldsymbol{\chi}_{\mathbf{u}_p},\partial_t\boldsymbol{\phi}_{\mathbf{u}_p};c_h) - a_p^d(\boldsymbol{\phi}_{\mathbf{u}_p},\partial_t\boldsymbol{\phi}_{\mathbf{u}_p};\partial_t c_h) - a_p^d(\boldsymbol{\chi}_{\mathbf{u}_p},\partial_t\boldsymbol{\phi}_{\mathbf{u}_p};\partial_t c_h) \\
&\quad - a_{BJS}(\partial_t\boldsymbol{\chi}_{\mathbf{u}_f},\partial_{tt}\boldsymbol{\chi}_{\boldsymbol{\eta}_p};\partial_t\boldsymbol{\phi}_{\mathbf{u}_f},\partial_{tt}\boldsymbol{\phi}_{\boldsymbol{\eta}_p}) - b_f(\partial_t\boldsymbol{\phi}_{\mathbf{u}_f},\partial_t\chi_{p_f}) - \alpha b_e(\partial_{tt}\boldsymbol{\phi}_{\boldsymbol{\eta}_p},\partial_t\chi_{p_p}) \\
&\quad + \alpha b_e(\partial_{tt}\boldsymbol{\chi}_{\boldsymbol{\eta}_p},\partial_t\phi_{p_p}) - \langle\partial_t\boldsymbol{\phi}_{\mathbf{u}_f}\cdot\mathbf{n}_f+\partial_{tt}\boldsymbol{\phi}_{\boldsymbol{\eta}_p}\cdot\mathbf{n}_p,\partial_t\chi_\lambda\rangle_{\Gamma_{fp}} - a_f(\partial_t\mathbf{u}_f,\partial_t\boldsymbol{\phi}_{\mathbf{u}_f};c-c_h) \\
&\quad - a_f(\mathbf{u}_f,\partial_t\boldsymbol{\phi}_{\mathbf{u}_f};\partial_t(c-c_h)) - a_p^d(\partial_t\mathbf{u}_p,\partial_t\boldsymbol{\phi}_{\mathbf{u}_p};c-c_h) - a_p^d(\mathbf{u}_p,\partial_t\boldsymbol{\phi}_{\mathbf{u}_p};\partial_t(c-c_h))\,.
\end{aligned}$$

Following the argument leading to (6.26), we deduce

$$\begin{aligned}
&\|\partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p}(t)\|^2_{\mathbf{H}^1(\Omega_p)} + s_0\|\partial_t\phi_{p_p}(t)\|^2_{L^2(\Omega_p)} + \int_0^t\left(\|\partial_t\boldsymbol{\phi}_{\mathbf{u}_f}\|^2_{\mathbf{H}^1(\Omega_f)} + \|\partial_t\boldsymbol{\phi}_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)}\right)ds \\
&\quad \le C\left(I_1 + \int_0^t\left(\|\partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p}\|^2_{\mathbf{H}^1(\Omega_p)} + \|E_c\|^2_{L^2(\Omega)} + \|\partial_t E_c\|^2_{L^2(\Omega)}\right)ds\right),
\end{aligned} \tag{6.39}$$

where $I_1$ is a collection of optimal order approximation and initial error terms. We next derive the bound for the transport equation. Differentiating in time the error equation (6.27) and taking $\psi_h=\partial_t\phi_c$ results in

$$(\phi\partial_{tt}\phi_c,\partial_t\phi_c) + (\mathbf{D}(\mathbf{u}_h)\partial_t\nabla\phi_c,\partial_t\nabla\phi_c) - (q^-\partial_t\phi_c,\partial_t\phi_c)$$

$$= -(\partial_t \mathbf{D}(\mathbf{u}_h)\nabla\phi_c, \partial_t\nabla\phi_c) - (\phi\partial_{tt}\chi_c, \partial_t\phi_c) - (\partial_t\mathbf{D}(\mathbf{u}_h)\nabla\chi_c, \partial_t\nabla\phi_c) - (\mathbf{D}(\mathbf{u}_h)\partial_t\nabla\chi_c, \partial_t\nabla\phi_c)$$
$$+ (\partial_t(\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}_h))\nabla c, \partial_t\nabla\phi_c) + ((\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}_h))\partial_t\nabla c, \partial_t\nabla\phi_c)$$
$$+ (\partial_t\phi_c\mathbf{u}_h, \partial_t\nabla\phi_c) + (\phi_c\partial_t\mathbf{u}_h, \partial_t\nabla\phi_c) + (\partial_t\chi_c\mathbf{u}_h, \partial_t\nabla\phi_c) + (\chi_c\partial_t\mathbf{u}_h, \partial_t\nabla\phi_c)$$
$$+ (\partial_t c\mathbf{E}_{\mathbf{u}}, \partial_t\nabla\phi_c) + (c\partial_t\mathbf{E}_{\mathbf{u}}, \partial_t\nabla\phi_c) + (q^-\partial_t\chi_c, \partial_t\phi_c)\,.$$

The key new term is $(\partial_t(\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}_h))\nabla c, \partial_t\nabla\phi_c)$. Since $\partial_t(\mathbf{D}(\mathbf{u}))_{i,j} = \sum_{k=1}^{d} \frac{\partial(\mathbf{D}(\mathbf{u}))_{i,j}}{\partial \mathbf{u}_k}\partial_t\mathbf{u}_k$, using (6.33), we have

$$|(\partial_t(\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{u}_h))\nabla c, \partial_t\nabla\phi_c)| \le C\left(\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} + \|\partial_t(\mathbf{u} - \mathbf{u}_h)\|_{\mathbf{L}^2(\Omega)}\right)\|\partial_t\nabla\phi_c\|_{L^2(\Omega)}\,.$$

Following the argument leading to (6.29), we obtain

$$\begin{aligned}
&\|\partial_t\phi_c(t)\|^2_{L^2(\Omega)} + \int_0^t \|\partial_t\nabla\phi_c\|^2_{\mathbf{L}^2(\Omega)}\,ds \\
&\quad \le C\left(I_2 + \int_0^t \left(\|\partial_t\phi_c\|^2_{L^2(\Omega)} + \|\mathbf{E}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t\mathbf{E}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)}\right)ds\right),
\end{aligned} \tag{6.40}$$

where $I_2$ is a collection of optimal order approximation and initial error terms. Combining (6.39), with a sufficiently small multiple of (6.40), (6.26), and a sufficiently small multiple of (6.29), and using Gronwall's inequality for the terms $\int_0^t \left(\|\partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p}\|^2_{\mathbf{H}^1(\Omega_p)} + \|\partial_t E_c\|^2_{L^2(\Omega)}\right)ds$ in (6.39) and $\int_0^t \|\partial_t\phi_c\|^2_{L^2(\Omega)}ds$ in (6.40), as well as $\int_0^t \left(\|\boldsymbol{\phi}_{\boldsymbol{\eta}_p}\|^2_{\mathbf{H}^1(\Omega_p)} + \|E_c\|^2_{L^2(\Omega)}\right)ds$ in (6.26) and $\int_0^t \|\phi_c\|^2_{L^2(\Omega)}ds$ in (6.29), results in

$$\begin{aligned}
&\|\partial_t\boldsymbol{\phi}_{\boldsymbol{\eta}_p}(t)\|^2_{\mathbf{H}^1(\Omega_p)} + s_0\|\partial_t\phi_{p_p}(t)\|^2_{L^2(\Omega_p)} + \|\partial_t\phi_c(t)\|^2_{L^2(\Omega)} \\
&\quad + \int_0^t \left(\|\partial_t\boldsymbol{\phi}_{\mathbf{u}_f}\|^2_{\mathbf{H}^1(\Omega_f)} + \|\partial_t\boldsymbol{\phi}_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} + \|\partial_t\nabla\phi_c\|^2_{\mathbf{L}^2(\Omega)}\right)ds \\
&\le C\,\exp(Lt)h^{2\min\{k_f, s_f+1, k_p+1, s_p+1, k_s, k_c\}}\,.
\end{aligned} \tag{6.41}$$

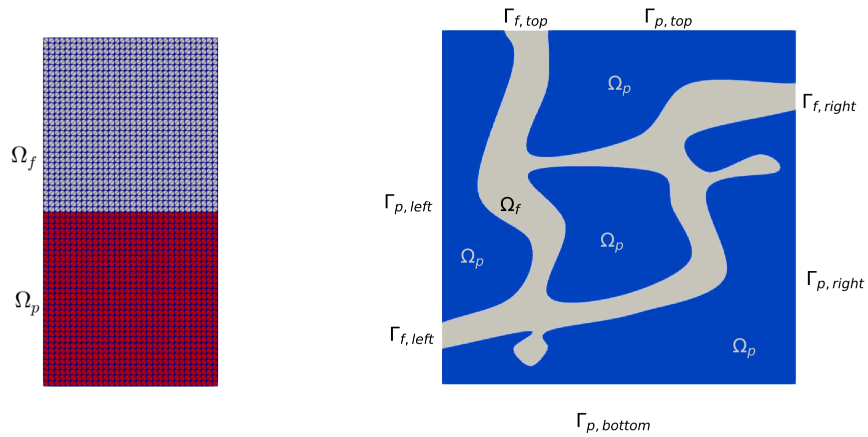
Bound (6.37) now follows from (6.38) and (6.41). □

# 7 Numerical results

In this section, we present two numerical results illustrating the performance of the fully discrete method associated with (3.4)–(3.5), using a backward Euler time discretization and a Picard iteration, implemented with the finite element package `FreeFEM` [40]. In the first test, we verify the theoretical convergence rates using a problem with a known analytical solution. The second example illustrates the application of the method to simulate flow and transport in a poroelastic medium with an irregular channel network and physically realistic parameters.

### Example 1: Convergence test

In this example, we present a convergence test verifying the theoretical convergence rates for a problem with a known analytical solution. Let $\Delta t$ denote the time step, $T = N\Delta t$ the final time,




(a) Convergence test (b) Flow in channelized media

Figure 7.1: Computational domains for Example 1 (left) and Example 2 (right).

$t_n = n\,\Delta t$, and $\varphi^n = \varphi(t_n)$. For a given spatial space $X$ with norm $\|\cdot\|_X$, we define the following discrete-in-time norms:

$$\|\phi\|_{l^2(0,T;X)} := \left(\Delta t \sum_{n=1}^{N} \|\phi^n\|_X^2\right)^{1/2}, \quad \|\phi\|_{l^\infty(0,T;X)} := \max_{1\le n\le N} \|\phi^n\|_X\,.$$

The computational domain is $\Omega = (0,1)\times(-1,1)$, with $\Omega_p = (0,1)\times(-1,0)$ and $\Omega_f = (0,1)\times(0,1)$, as shown in Figure 7.1(a). The manufactured solution is

$$\mathbf{u}_f = \exp(t)\sin(\pi x)\begin{pmatrix}\cos(\pi y)\\ y\end{pmatrix}, \quad p_f = \exp(t)\sin(\pi x)\cos\left(\frac{\pi y}{2}\right) + 2\,\pi\cos(\pi t)\,,$$

$$\mathbf{u}_p = \frac{\pi\exp(t)}{\mu(c)}\begin{pmatrix}-\cos(\pi x)\cos\left(\dfrac{\pi y}{2}\right)\\ \dfrac{1}{2}\sin(\pi x)\sin\left(\dfrac{\pi y}{2}\right)\end{pmatrix}, \quad p_p = \exp(t)\sin(\pi x)\cos\left(\frac{\pi y}{2}\right),$$

$$\boldsymbol{\eta}_p = \sin(\pi t)\begin{pmatrix}-3x+\cos(y)\\ y+1\end{pmatrix}, \quad c = \frac{t}{\pi}\left(\cos(\pi x)+\cos(\pi y)\right), \quad \text{and} \quad \mu(c) = 2 - \frac{1}{c+1}\,.$$

In addition, we set $K = \mathbf{I}$, and the diffusion-dispersion parameters in (2.9) are $\phi = 1$ in $\Omega_f$, $\phi = 0.4$ in $\Omega_p$, $d_m = 10^{-3}$, and $\alpha_l = \alpha_t = 1$. On the interface $\Gamma_{fp} = y = 0$, we set the constant viscosity $\mu(c_0) = \mu(c(0)) = 1$. The manufactured Stokes–Biot solution does not satisfy the interface conditions in (2.7). Consequently, the right-hand side must be modified to account for the resulting interface residuals. Furthermore, the data $\mathbf{f}_f$, $q_f$, $\mathbf{f}_p$, and $q_p$ are computed from (2.1)–(2.2) using the exact solution. The model is complemented with appropriate Dirichlet boundary conditions for $\mathbf{u}_f$, $\mathbf{u}_p$, $\boldsymbol{\eta}_p$, and $c$, as well as the initial datum $c_0$. The initial values $p_{p,0}$ and $\boldsymbol{\eta}_{p,0}$ are obtained by solving (3.18)–(3.19).

We investigate convergence for two choices of finite element spaces. For the lower-order choice, we use a MINI-element-based discretization, namely $\mathcal{P}_1^b - \mathcal{P}_1$ for Stokes, lowest-order Raviart–Thomas spaces $\mathcal{RT}_0 - \mathcal{P}_0$ for Darcy, $\mathcal{P}_1$ elements for displacement, $\mathcal{P}_0$ elements for the Lagrange multiplier, and $\mathcal{P}_1$ elements for concentration. The higher-order choice consists of a Taylor–Hood-based discretization with $\mathcal{P}_2 - \mathcal{P}_1$, $\mathcal{RT}_1 - \mathcal{P}_1^{dc}$, $\mathcal{P}_2$, $\mathcal{P}_1^{dc}$, and $\mathcal{P}_2$, respectively. The final time is $T = 10^{-3}$ with $\Delta t = 10^{-4}$ for the MINI-element discretization, and $T = 10^{-5}$ with $\Delta t = 10^{-6}$ for the Taylor–Hood discretization. In both cases, $\Delta t$ is chosen sufficiently small so that the time discretization error does not affect the spatial convergence rates. The relative errors and convergence rates for a sequence of mesh refinements are reported in Tables 7.1 and 7.2 for the lower and higher order choices, respectively. We observe first-order convergence for all variables in Table 7.1 and second-order convergence in Table 7.2, in agreement with Theorems 6.1 and 6.3.

| $h$ | $\|\mathbf{E}_{\mathbf{u}_f}\|_{l^2(\mathbf{H}^1(\Omega_f))}$ error | rate | $\|E_{p_f}\|_{l^2(L^2(\Omega_f))}$ error | rate | $\|\mathbf{E}_{\mathbf{u}_p}\|_{l^2(\mathbf{L}^2(\Omega_p))}$ error | rate | $\|\nabla\cdot\mathbf{E}_{\mathbf{u}_p}\|_{l^2(L^2(\Omega_p))}$ error | rate | $\|E_{p_p}\|_{l^\infty(L^2(\Omega_p))}$ error | rate |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.354 | 3.27E-02 | – | 5.36E-02 | – | 1.16E-02 | – | 3.99E-02 | – | 1.03E-01 | – |
| 0.177 | 1.54E-02 | 1.091 | 1.46E-02 | 1.877 | 5.81E-03 | 1.009 | 2.01E-02 | 0.988 | 5.17E-02 | 0.990 |
| 0.088 | 7.52E-03 | 1.031 | 4.62E-03 | 1.659 | 2.90E-03 | 1.000 | 1.01E-02 | 0.997 | 2.59E-02 | 0.997 |
| 0.044 | 3.73E-03 | 1.013 | 1.55E-03 | 1.576 | 1.45E-03 | 1.000 | 5.05E-03 | 0.999 | 1.29E-02 | 0.999 |
| 0.022 | 1.86E-03 | 1.006 | 5.33E-04 | 1.541 | 7.26E-04 | 1.000 | 2.52E-03 | 1.000 | 6.47E-03 | 1.000 |
| 0.011 | 9.26E-04 | 1.003 | 1.85E-04 | 1.522 | 3.63E-04 | 1.000 | 1.26E-03 | 1.000 | 3.24E-03 | 1.000 |

| $\|\mathbf{E}_s\|_{l^\infty(\mathbf{H}_1(\Omega_p))}$ error | rate | $\|E_c\|_{l^2(H^1(\Omega))}$ error | rate | $\|E_c\|_{l^\infty(L^2(\Omega))}$ error | rate | $h_{fp}$ | $\|E_\lambda\|_{l^2(L^2(\Gamma_{fp}))}$ error | rate | iter |
|---|---|---|---|---|---|---|---|---|---|
| 1.14E-02 | – | 6.72E-06 | – | 1.72E-05 | – | 1/4 | 5.18E-03 | – | 3 |
| 4.19E-03 | 1.441 | 3.25E-06 | 1.046 | 3.57E-06 | 2.267 | 1/8 | 2.55E-03 | 1.024 | 3 |
| 1.17E-03 | 1.835 | 1.60E-06 | 1.026 | 8.08E-07 | 2.144 | 1/16 | 1.27E-03 | 1.006 | 3 |
| 3.00E-04 | 1.968 | 7.93E-07 | 1.010 | 1.94E-07 | 2.056 | 1/32 | 6.34E-04 | 1.001 | 3 |
| 7.56E-05 | 1.988 | 3.96E-07 | 1.002 | 4.86E-08 | 2.000 | 1/64 | 3.17E-04 | 1.000 | 3 |
| 1.95E-05 | 1.953 | 1.98E-07 | 0.998 | 1.25E-08 | 1.959 | 1/128 | 1.59E-04 | 1.000 | 3 |

Table 7.1: [Example 1] Mesh sizes, errors, convergence rates, and average number of Picard iterations for the MINI-element-based discretization of the Stokes–Biot–transport model with $\Delta t = 10^{-4}$.

### Example 2: Flow and transport through poroelastic media with channel network

In this example, we apply our method to a physically realistic flow and transport problem in a poroelastic aquifer containing an irregularly shaped network of channels. The computational domain, of size $2\,\mathrm{m} \times 2\,\mathrm{m}$ and representing a horizontal cross-section of the aquifer, is shown in Figure 7.1(b). The channel network corresponds to the free-fluid region $\Omega_f$, while the remainder of the domain constitutes the poroelastic region $\Omega_p$. The flow is driven from left to right by a pressure drop of 1 kPa. The transport equation models the concentration of mercury, which enters the water-filled domain through the inflow boundary. For the nonlinear viscosity function $\mu(c)$, we employ the well-established quarter-power mixing law [41] :

$$\mu(c) = \Big(c\,\mu_s^{-0.25} + (1-c)\,\mu_0^{-0.25}\Big)^{-4},$$

where $\mu_s$ is the viscosity of the solute (mercury) and $\mu_0$ is the viscosity of the solvent (water). For the poroelastic structure, we modify the first equation in (2.2a) by adding a spring-like term (see [14, 15]):

$$\boldsymbol{\eta}_p - \nabla\cdot\boldsymbol{\sigma}_p(\boldsymbol{\eta}_p, p_p) = 0 \quad \text{in} \quad \Omega_p \times (0,T]\,. \tag{7.1}$$

| $h$ | $\|\mathbf{E}_{\mathbf{u}_f}\|_{l^2(\mathbf{H}^1(\Omega_f))}$ error | rate | $\|E_{p_f}\|_{l^2(L^2(\Omega_f))}$ error | rate | $\|\mathbf{E}_{\mathbf{u}_p}\|_{l^2(\mathbf{L}^2(\Omega_p))}$ error | rate | $\|\nabla\cdot\mathbf{E}_{\mathbf{u}_p}\|_{l^2(L^2(\Omega_p))}$ error | rate | $\|E_{p_p}\|_{l^\infty(L^2(\Omega_p))}$ error | rate |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.354 | 4.35E-04 | – | 9.99E-05 | – | 9.82E-05 | – | 4.63E-04 | – | 1.19E-02 | – |
| 0.177 | 1.12E-04 | 1.963 | 1.57E-05 | 2.672 | 2.49E-05 | 1.982 | 1.17E-04 | 1.989 | 2.99E-03 | 1.990 |
| 0.088 | 2.81E-05 | 1.988 | 3.27E-06 | 2.259 | 6.25E-06 | 1.993 | 2.92E-05 | 1.997 | 7.49E-04 | 1.998 |
| 0.044 | 7.05E-06 | 1.996 | 7.78E-07 | 2.073 | 1.57E-06 | 1.997 | 7.31E-06 | 1.999 | 1.87E-04 | 1.999 |
| 0.022 | 1.77E-06 | 1.999 | 1.92E-07 | 2.019 | 3.92E-07 | 1.999 | 1.83E-06 | 2.000 | 4.68E-05 | 2.000 |
| 0.011 | 4.41E-07 | 1.999 | 4.78E-08 | 2.005 | 9.80E-08 | 1.999 | 4.57E-07 | 2.000 | 1.17E-05 | 2.000 |

| $\|\mathbf{E}_s\|_{l^\infty(\mathbf{H}_1(\Omega_p))}$ error | rate | $\|E_c\|_{l^2(H^1(\Omega))}$ error | rate | $\|E_c\|_{l^\infty(L^2(\Omega))}$ error | rate | $h_{fp}$ | $\|E_\lambda\|_{l^2(L^2(\Gamma_{fp}))}$ error | rate | iter |
|---|---|---|---|---|---|---|---|---|---|
| 1.59E-03 | – | 6.50E-10 | – | 1.11E-08 | – | 1/4 | 5.12E-05 | – | 2 |
| 2.78E-04 | 2.519 | 1.61E-10 | 2.010 | 1.43E-09 | 2.955 | 1/8 | 1.28E-05 | 1.996 | 2 |
| 4.90E-05 | 2.504 | 4.00E-11 | 2.012 | 1.81E-10 | 2.981 | 1/16 | 3.21E-06 | 1.999 | 2 |
| 8.66E-06 | 2.501 | 9.98E-12 | 2.004 | 2.27E-11 | 2.994 | 1/32 | 8.03E-07 | 2.000 | 2 |
| 1.53E-06 | 2.501 | 2.50E-12 | 1.999 | 2.85E-12 | 2.995 | 1/64 | 2.01E-07 | 2.000 | 2 |
| 2.71E-07 | 2.500 | 6.33E-13 | 1.980 | 3.68E-13 | 2.955 | 1/128 | 5.02E-08 | 2.000 | 2 |

Table 7.2: [Example 1] Mesh sizes, errors, convergence rates, and average number of Picard iterations for the Taylor–Hood-element-based discretization of the Stokes–Biot–transport model with $\Delta t = 10^{-6}$.

The additional term accounts for circumferential strain around the channel, which is lost in the two-dimensional setting. We note that it does not alter the theoretical results, as it contributes a positive term to the weak formulation. The model parameters, which are physically realistic for an aquifer, are given in Table 7.3. We note the very small values of the storativity and permeability and the very large values of the Lamé coefficients, which results in a challenging computational system.

The boundary conditions are as follows:

$$p_p = 1 \quad \text{on} \quad \Gamma_{p,left}\,, \quad p_p = 0 \quad \text{on} \quad \Gamma_{p,right}\,, \quad \mathbf{u}_p\cdot\mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_{p,top}\cup\Gamma_{p,bottom}\,;$$

$$\boldsymbol{\sigma}_p\mathbf{n}_p = \mathbf{0} \quad \text{on} \quad \Gamma_{p,left}\,, \quad \boldsymbol{\eta}_p = \mathbf{0} \quad \text{on} \quad \Gamma_{p,right}\,;$$

$$(\boldsymbol{\sigma}_p\mathbf{n}_p)\cdot\mathbf{n}_p = 0 \quad \text{and} \quad \boldsymbol{\eta}_p\cdot\boldsymbol{\tau}_p = 0 \quad \text{on} \quad \Gamma_{p,top}\cup\Gamma_{p,bottom}\,;$$

$$\mathbf{u}_f\cdot\mathbf{n}_f = 0.2 \quad \text{and} \quad \mathbf{u}_f\cdot\boldsymbol{\tau}_f = 0 \quad \text{on} \quad \Gamma_{f,left}\,;$$

$$(\boldsymbol{\sigma}_f\mathbf{n}_f)\cdot\mathbf{n}_f = 0 \quad \text{and} \quad \mathbf{u}_f\cdot\boldsymbol{\tau}_f = 0 \quad \text{on} \quad \Gamma_{f,right}\cup\Gamma_{f,top}\,;$$

$$(c\mathbf{u} - \mathbf{D}\nabla c)\cdot\mathbf{n} = (c_{in}\mathbf{u})\cdot\mathbf{n}, \quad c_{in} = 1 \quad \text{on} \quad \Gamma_{p,left}\,, \quad (\mathbf{D}\nabla c)\cdot\mathbf{n} = 0, \qquad \text{on} \quad \partial\Omega\backslash\Gamma_{p,left}\,.$$

We set the initial conditions as $\boldsymbol{\eta}_{p,0} = \mathbf{0}$, $p_{p,0} = 0$, and $c_0 = 0$.

The total simulation time is $T = 25\,\mathrm{s}$, with a time-step size of $\Delta t = 0.01\,\mathrm{s}$. For the spatial discretization, we employ the MINI-element-based approach, namely, $\mathcal{P}_1^b$–$\mathcal{P}_1$ elements for the Stokes problem, the lowest-order Raviart–Thomas pair $\mathcal{RT}_0 - \mathcal{P}_0$ for the Darcy problem, $\mathcal{P}_1$ elements for the displacement, $\mathcal{P}_0$ elements for the Lagrange multiplier, and $\mathcal{P}_1$ elements for the concentration. Similar results are obtained with the Taylor–Hood-based discretization.

Figure 7.2 shows the computed Darcy pressure, displacement, and Stokes velocity fields at the final time $T = 25\,\mathrm{s}$. The Darcy velocity $\mathbf{u}_p$ and pressure $p_p$ are largest in the region between the

| Parameter | Units | Symbol | Value |
|---|---|---|---|
| Mass storativity | $\mathrm{kPa}^{-1}$ | $s_0$ | $5 \times 10^{-2}$ |
| Lamé coeff. | kPa | $\lambda_p$ | $5/18 \times 10^7$ |
| Lamé coeff. | kPa | $\mu_p$ | $5/12 \times 10^7$ |
| Permeability | $\mathrm{m}^2$ | $K$ | $10^{-8}\,\mathbf{I}$ |
| Biot-Willis constant | | $\alpha$ | 1 |
| BJS coeff. | | $\alpha_{BJS}$ | 1 |
| Solute viscosity | kPa s | $\mu_s$ | $1.526 \times 10^{-6}$ |
| Solvent viscosity | kPa s | $\mu_0$ | $8.9 \times 10^{-7}$ |
| Porosity (in $\Omega_f$) | | $\phi$ | 1.0 |
| Porosity (in $\Omega_p$) | | $\phi$ | 0.4 |
| diffusivity (in $\Omega_f$) | $\mathrm{m}^2/\mathrm{s}$ | $d_m$ | $10^{-6}$ |
| diffusivity (in $\Omega_p$) | $\mathrm{m}^2/\mathrm{s}$ | $d_m$ | $5 \times 10^{-4}$ |
| longit. dispersion | m | $\alpha_l$ | $5 \times 10^{-4}$ |
| transv. dispersion | m | $\alpha_t$ | $5 \times 10^{-4}$ |

Table 7.3: Physical parameters for Example 2.

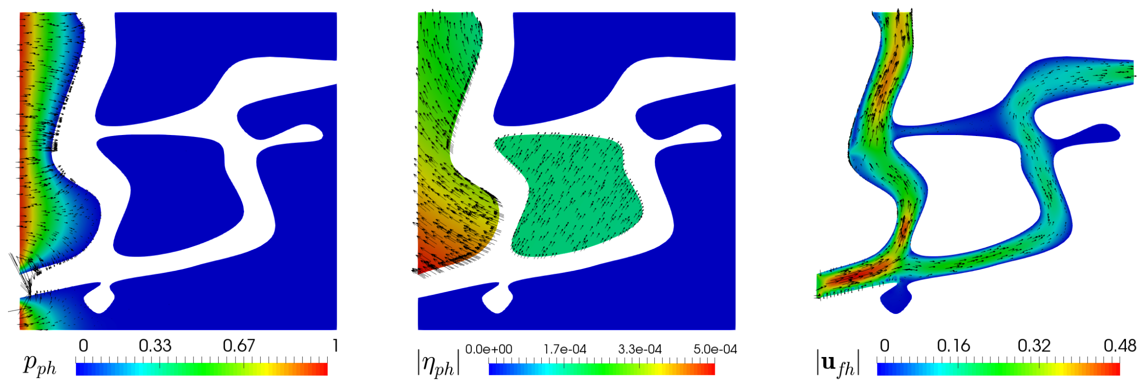



Figure 7.2: [Example 2] Computed Darcy pressure (with arrows scaled according to the Darcy velocity), displacement, and Stokes velocity fields at the final time $T = 25$ seconds.

inflow boundary and the adjacent channel, which leads to larger displacements $\boldsymbol{\eta}_p$ in this region. We observe that, at each bifurcation of the channel network, the Stokes velocity field $\mathbf{u}_f$ follows a preferential path through the wider channel. Six snapshots of the concentration solution at different times are shown in Figure 7.3. As expected, the mercury follows the flow and propagates much faster in the channel network than in the poroelastic region. On the other hand, we also observe mercury propagation in the inflow-side poroelastic region, where the Darcy velocity is larger, causing the solute to enter the left channel through the fluid–poroelastic interface. This results in two coexisting mercury streams in the channel network. The two streams do not mix until close to the final simulation time. In Figure 7.4, we show the fluid viscosity $\mu(c_h)$ at the same six snapshots. Its profile is similar to that of the concentration. The lower bound of the color scale in Figure 7.4 (blue) corresponds to the viscosity of water, while the upper bound (red) corresponds to the viscosity

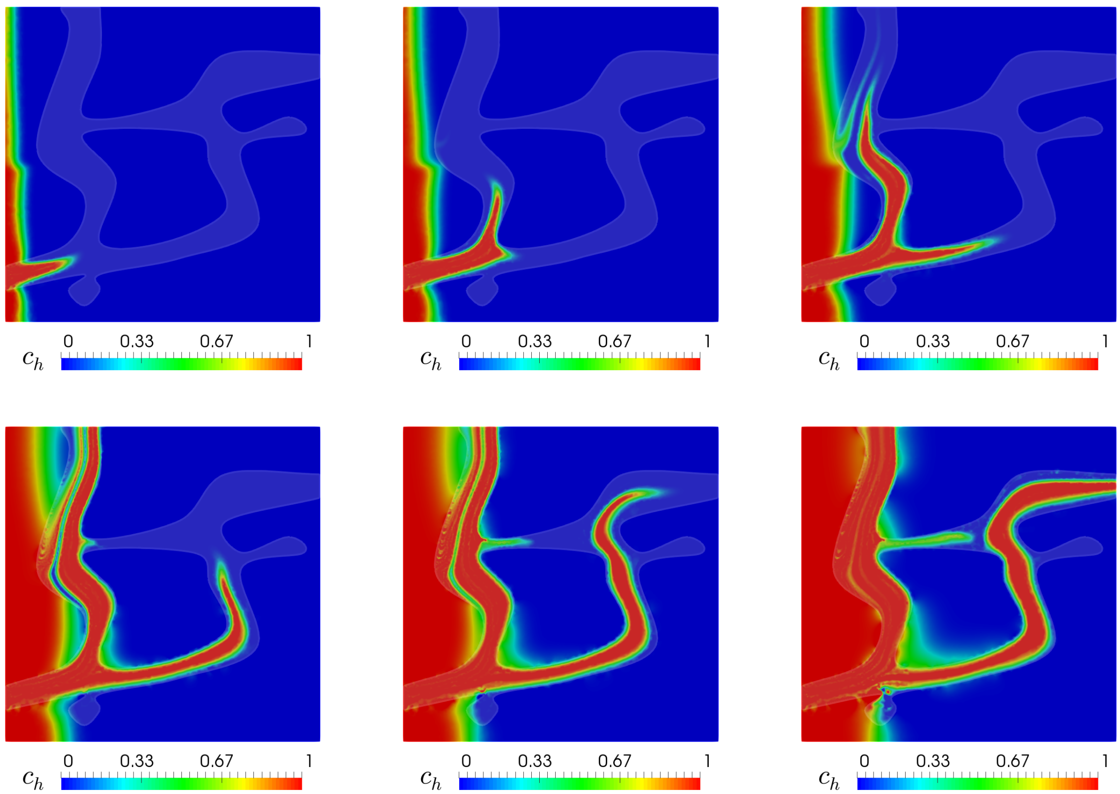



Figure 7.3: [Example 2] Computed concentration at times $T \in \{1, 2.5, 5\}$ seconds (top row) and $T \in \{10, 15, 25\}$ seconds (bottom row).

of mercury. This example illustrates the ability of the proposed numerical method to simulate a challenging problem with irregular geometry and physically realistic parameters that vary over a wide range. The obtained results are free of numerical instabilities and are qualitatively consistent with the expected physical behavior. In addition, several interesting high-fidelity transport patterns are observed.

# 8 Conclusions

In this paper we developed a mathematical model and its numerical approximation for coupling fluid–poroelastic structure interaction with transport. The model is fully coupled and nonlinear, which presents difficulties in the analysis. To handle this, we first analyze the decoupled and linearized subproblems and then employ a contraction argument for an iterative procedure to obtain a solution of the coupled nonlinear problem. The well posedness analysis of the proposed semidiscrete continuous-in-time finite element scheme utilizes the framework of the continuous problem. In addition, we derive optimal order error estimates for all variables in their corresponding norms. The presented numerical experiments verify the theoretical results and illustrate the good performance of the method for a computationally challenging problem with physically realistic parameters and irregular geometry. Future research directions include studying the Navier–Stokes–Biot–transport problem and utilizing Banach space techniques [38] to handle lower velocity regularity.

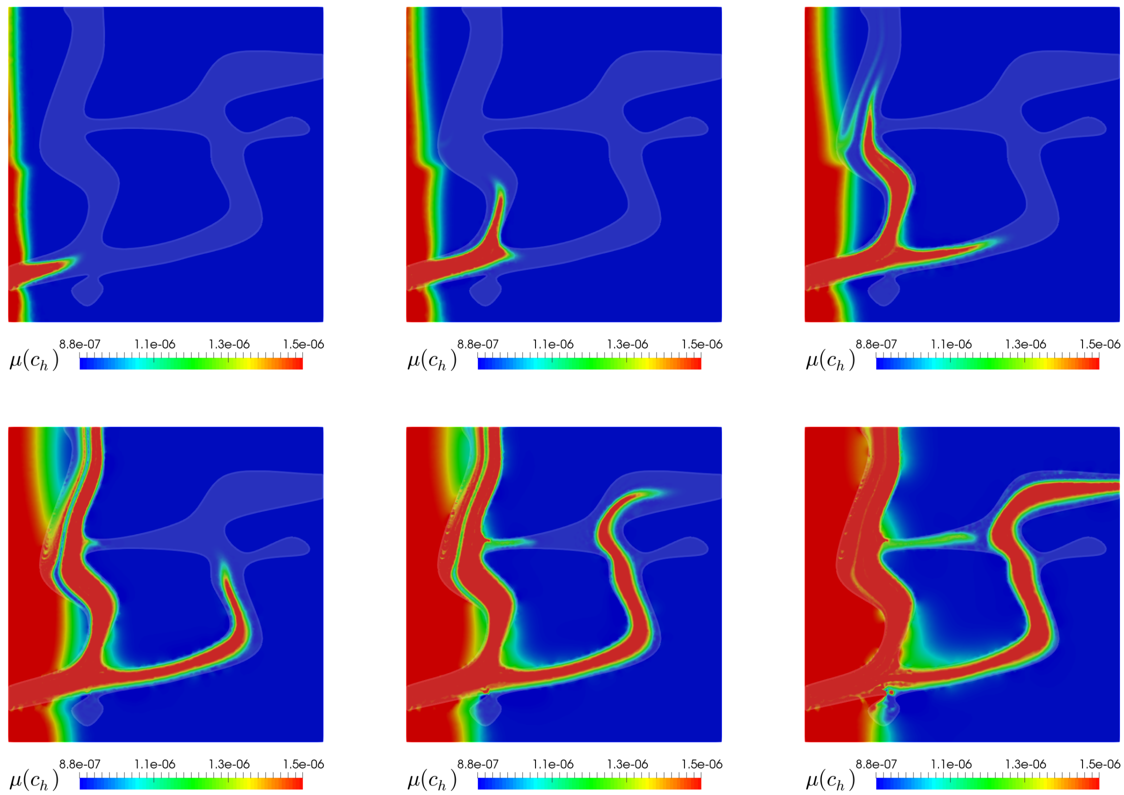



Figure 7.4: [Example 2] Computed fluid viscosity at times $T \in \{1, 2.5, 5\}$ seconds (top row) and $T \in \{10, 15, 25\}$ seconds (bottom row).

## References


[1] I. Ambartsumyan, V. J. Ervin, T. Nguyen, and I. Yotov. A nonlinear Stokes–Biot model for the interaction of a non-Newtonian fluid with poroelastic media. *ESAIM Math. Model. Numer. Anal.*, 53(6):1915–1955, 2019.

[2] I. Ambartsumyan, E. Khattatov, T. Nguyen, and I. Yotov. Flow and transport in fractured poroelastic media. *GEM Int. J. Geomath.*, 10(1):1–34, 2019.

[3] I. Ambartsumyan, E. Khattatov, I. Yotov, and P. Zunino. A Lagrange multiplier method for a Stokes-Biot fluid-poroelastic structure interaction model. *Numer. Math.*, 140(2):513–553, 2018.

[4] L. Badea, M. Discacciati, and A. Quarteroni. Numerical analysis of the Navier-Stokes/Darcy coupling. *Numer. Math.*, 115(2):195–227, 2010.

[5] S. Badia, A. Quaini, and A. Quarteroni. Coupling Biot and Navier-Stokes equations for modelling fluid-poroelastic media interaction. *J. Comput. Phys.*, 228(21):7986–8014, 2009.

[6] Y. Bazilevs, K. Takizawa, and T. E. Tezduyar. *Computational fluid-structure interaction: methods and applications*. John Wiley & Sons, 2013.

[7] L. Bociu, S. Canic, B. Muha, and J. T. Webster. Multilayered poroelasticity interacting with Stokes flow. *SIAM J. Math. Anal.*, 53(6):6243–6279, 2021.

[8] D. Boffi, F. Brezzi, and M. Fortin. *Mixed finite element methods and applications*, volume 44 of *Springer Series in Computational Mathematics*. Springer, Heidelberg, 2013.

[9] W. M. Boon, M. Hornkjø l, M. Kuchta, K.-A. Mardal, and R. Ruiz-Baier. Parameter-robust methods for the Biot-Stokes interfacial coupling without Lagrange multipliers. *J. Comput. Phys.*, 467:Paper No. 111464, 25, 2022.

[10] K. E. Brenan, S. L. Campbell, and L. R. Petzold. *Numerical solution of initial-value problems in differential-algebraic equations.* SIAM, 1995.

[11] M. K. Brun, E. Ahmed, I. Berre, J. M. Nordbotten, and F. A. Radu. Monolithic and splitting solution schemes for fully coupled quasi-static thermo-poroelasticity with nonlinear convective transport. *Comput. Math. Appl.*, 80(8):1964–1984, 2020.

[12] M. K. Brun, E. Ahmed, J. M. Nordbotten, and F. A. Radu. Well-posedness of the fully coupled quasi-static thermo-poroelastic equations with nonlinear convective transport. *J. Math. Anal. Appl.*, 471(1-2):239–266, 2019.

[13] M. Bukač, S. Čanić, B. Muha, and Y. Wang. A computational algorithm for optimal design of a bioartificial organ scaffold architecture. *PLoS Comput. Biol.*, 20(11):e1012079, 2024.

[14] M. Bukač, I. Yotov, R. Zakerzadeh, and P. Zunino. Partitioning strategies for the interaction of a fluid with a poroelastic material based on a Nitsche's coupling approach. *Comput. Methods Appl. Mech. Engrg.*, 292:138–170, 2015.

[15] M. Bukač, I. Yotov, and P. Zunino. An operator splitting approach for the interaction between a fluid and a multilayered poroelastic structure. *Numer. Methods Partial Differential Equations*, 31(4):1054–1100, 2015.

[16] M. Bukač, I. Yotov, and P. Zunino. Dimensional model reduction for flow through fractures in poroelastic media. *ESAIM Math. Model. Numer. Anal.*, 51(4):1429–1471, 2017.

[17] H.-J. Bungartz and M. Schäfer. *Fluid-structure interaction: modelling, simulation, optimisation*, volume 53. Springer Science & Business Media, 2006.

[18] J. Careaga, G. N. Gatica, C. Inzunza, and R. Ruiz-Baier. New Banach spaces-based mixed finite element methods for the coupled poroelasticity and heat equations. *IMA J. Numer. Anal.*, 45(4):1936–1984, 2025.

[19] S. Caucao, A. Dalal, and I. Yotov. A Banach space formulation for the fully dynamic Navier-Stokes-Biot coupled problem. https://doi.org/10.48550/arXiv.2410.06322.

[20] S. Caucao, T. Li, and I. Yotov. A multipoint stress-flux mixed finite element method for the Stokes-Biot model. *Numer. Math.*, 152(2):411–473, 2022.

[21] A. Çeşmelioğlu. Analysis of the coupled Navier-Stokes/Biot problem. *J. Math. Anal. Appl.*, 456(2):970–991, 2017.

[22] A. Cesmelioglu and P. Chidyagwai. Numerical analysis of the coupling of free fluid with a poroelastic material. *Numer. Methods Partial Differential Equations*, 36(3):463–494, 2020.

[23] A. Cesmelioglu, J. J. Lee, and S. Rhebergen. Hybridizable discontinuous Galerkin methods for the coupled Stokes-Biot problem. *Comput. Math. Appl.*, 144:12–33, 2023.

[24] A. Cesmelioglu, J. J. Lee, and S. Rhebergen. A hybridizable discontinuous Galerkin method for the coupled Navier-Stokes/Biot problem. *ESAIM Math. Model. Numer. Anal.*, 58(4):1461–1495, 2024.

[25] A. Cesmelioglu, D. D. Pham, and S. Rhebergen. A hybridizable discontinuous Galerkin method for the fully coupled time-dependent Stokes/Darcy-transport problem. *ESAIM Math. Model. Numer. Anal.*, 57(3):1257–1296, 2023.

[26] A. Çeşmelioğlu and B. Rivière. Existence of a weak solution for the fully coupled Navier-Stokes/Darcy-transport problem. *J. Differential Equations*, 252(7):4138–4175, 2012.

[27] P. G. Ciarlet. *The finite element method for elliptic problems*, volume 40 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002.

[28] P. G. Ciarlet. *Linear and nonlinear functional analysis with applications.* Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013.

[29] E. A. Coddington and N. Levinson. *Theory of ordinary differential equations.* McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[30] A. Dalal, R. Durst, A. Quaini, and I. Yotov. A Robin-Robin splitting method for the Stokes-Biot fluid-poroelastic structure interaction model. *Math. Comp.*, 2025; https://doi.org/10.1090/mcom/4156.

[31] M. Discacciati, E. Miglio, and A. Quarteroni. Mathematical and numerical models for coupling surface and groundwater flows. *Appl. Numer. Math.*, 43(1-2):57–74, 2002. 19th Dundee Biennial Conference on Numerical Analysis (2001).

[32] J. Douglas, Jr., R. E. Ewing, and M. F. Wheeler. The approximation of the pressure by a mixed method in the simulation of miscible displacement. *RAIRO Anal. Numér.*, 17(1):17–33, 1983.

[33] V. Ervin, E. Jenkins, and S. Sun. Coupled generalized nonlinear Stokes flow with flow through a porous medium. *SIAM J. Numer. Anal.*, 47(2):929–952, 2009.

[34] L. C. Evans. *Partial Differential Equations, second ed.*, volume 19. American Mathematical Society, Providence, Rhode Island, USA, 2010.

[35] X. Feng. On existence and uniqueness results for a coupled system modeling miscible displacement in porous media. *J. Math. Anal. Appl.*, 194(3):883–910, 1995.

[36] J. Galvis and M. Sarkis. Non-matching mortar discretization analysis for the coupling Stokes-Darcy equations. *Electron. Trans. Numer. Anal.*, 26:350–384, 2007.

[37] G. N. Gatica. *A Simple Introduction to the Mixed Finite Element Method. Theory and Applications.* Springer Briefs in Mathematics. Springer, Cham, 2014.

[38] G. N. Gatica, S. Meddahi, and R. Ruiz-Baier. An $L^p$ spaces-based formulation yielding a new fully mixed finite element method for the coupled Darcy and heat equations. *IMA J. Numer. Anal.*, 42(4):3154–3206, 2022.

[39] V. Girault and B. Rivière. DG approximation of coupled Navier-Stokes and Darcy equations by Beaver-Joseph-Saffman interface condition. *SIAM J. Numer. Anal.*, 47(3):2052–2089, 2009.

[40] F. Hecht. New development in FreeFem++. *J. Numer. Math.*, 20(3-4):251–265, 2012.

[41] E. Koval. A method for predicting the performance of unstable miscible displacement in heterogeneous media. *Soc. Pet. Eng. J.*, 3(02):145–154, 1963.

[42] W. J. Layton, F. Schieweck, and I. Yotov. Coupling fluid flow with porous media flow. *SIAM J. Numer. Anal.*, 40(6):2195–2218 (2003), 2002.

[43] T. Li, S. Caucao, and I. Yotov. An augmented fully mixed formulation for the quasistatic Navier-Stokes-Biot model. *IMA J. Numer. Anal.*, 44(2):1153–1210, 2024.

[44] T. Li and I. Yotov. A mixed elasticity formulation for fluid–poroelastic structure interaction. *ESAIM Math. Model. Numer. Anal.*, 56(1):01–40, 2022.

[45] T. Richter. *Fluid-structure Interactions: Models, Analysis and Finite Elements*, volume 118. Springer, 2017.

[46] B. Riviére and I. Yotov. Locally conservative coupling of Stokes and Darcy flows. *SIAM J. Numer. Anal.*, 42(5):1959–1977, 2005.

[47] H. L. Royden and P. Fitzpatrick. *Real analysis*, volume 32. Macmillan New York, 1988.

[48] H. Rui and J. Zhang. A stabilized mixed finite element method for coupled Stokes and Darcy flows with transport. *Comput. Methods Appl. Mech. Engrg.*, 315:169–189, 2017.

[49] R. Ruiz-Baier, M. Taffetani, H. D. Westermeyer, and I. Yotov. The Biot-Stokes coupling using total pressure: formulation, analysis and application to interfacial flow in the eye. *Comput. Methods Appl. Mech. Engrg.*, 389:Paper No. 114384, 30, 2022.

[50] L. R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, 1990.

[51] A. Seboldt, O. Oyekole, J. Tambača, and M. Bukač. Numerical modeling of the fluid-porohyperelastic structure interaction. *SIAM J. Sci. Comput.*, 43(4):A2923–A2948, 2021.

[52] R. E. Showalter. Poroelastic filtration coupled to Stokes flow. In *Control theory of partial differential equations*, volume 242 of *Lect. Notes Pure Appl. Math.*, pages 229–241. Chapman & Hall/CRC, Boca Raton, FL, 2005.

[53] D. Vassilev and I. Yotov. Coupling Stokes-Darcy flow with transport. *SIAM J. Sci. Comput.*, 31(5):3661–3684, 2009.

[54] S. Čanić, Y. Wang, and M. Bukač. A next-generation mathematical model for drug-eluting stents. *SIAM J. Appl. Math.*, 81(4):1503–1529, 2021.

[55] X. Wang and I. Yotov. A Lagrange multiplier formulation for the fully dynamic Navier–Stokes–Biot system. *IMA J. Numer. Anal.*, 2025; https://doi.org/10.1093/imanum/draf074.

[56] J. Zhang, H. Rui, and Y. Cao. A partitioned method with different time steps for coupled Stokes and Darcy flows with transport. *Int. J. Numer. Anal. Model.*, 16(3):463–498, 2019.